SHADOW PRICE APPROACH TO PRODUCTIVITY MEASUREMENT

A Modified Malmquist Index

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ABSTRACT
The Fisher ideal total factor productivity (TFP) index requires complete quantity and price information. If input-output quantities are allocatively efficient, the Fisher index can be approximated by the Malmquist TFP index, which does not require price information. This paper elaborates on the conditions under which the Malmquist and Fisher indexes are equivalent. Drawing from these insights, we develop a modification of the Malmquist approach to better approximate the 'superlative' Fisher index with quantity data only. We illustrate the new approach by an empirical application to aggregate data of 14 OECD countries.
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1. INTRODUCTION

Productivity changes at the macro level of nations and industries as well as the micro level of firms and plants is of obvious interest to economists. Productivity change is defined as the ratio of change in output to change in input, and it encompasses such factors as 1) technological progress/regress, 2) operational efficiency, and 3) utilization of economies of scale and specialization. Measuring the degree of productivity change, and explaining the measured changes by changes of these underlying factors are the main concerns of the productivity analysis.

In the elementary case where only a single output is produced by employing a single input, measuring the productivity change is a trivially simple undertaking. However, virtually all production units produce multiple outputs and/or consume multiple inputs. The fundamental challenge in measuring *Total Factor Productivity* (hereafter TFP) changes comes from the need to aggregate the various inputs and outputs.

Many alternative approaches of aggregating inputs and outputs are available (see e.g. Diewert, 1992, 2000, for discussion). However, there seems to be no question about the need to account for the *values* of the inputs and outputs; expensive or important commodities should be assigned a greater weight than inexpensive or unimportant ones. Conventional index theory typically uses economic prices (or cost/revenue shares) as weights. A prime example is the Fisher TFP index. Diewert (1992) considered 20 different properties that any productivity index should have, and showed that the Fisher ideal index has all suggested properties, outperforming any other candidate index.

Unfortunately, the Fisher ideal index cannot be computed if the economically relevant prices (or cost/revenue shares) are not known. For many commodities, monetary prices do not exist (free commodities, intangible assets, new commodities introduced in the target period), or the prices are biased due to market failures (monopoly power, externalities) or governmental interference (tariffs, taxation, subsidizing, regulation). Moreover, it can often be difficult to obtain reliable price data for research purposes even from commodities that do go with the price tag. For example, pricing the services of durable inputs (capital goods) that are used over several time periods is a well-recognized problem.

In these circumstances, a frequently employed alternative is the Malmquist TFP index (introduced by Caves, Christensen, and Diewert, 1982a,b), which does not require any price data whatsoever. Interestingly, under some quite general conditions the Malmquist TFP index can approximate the Fisher ideal index, as pointed out by Diewert (1992), Färe and Grosskopf (1992), and Balk (1993). The approximation essentially relies on recovering price information from the shadow prices of the observed quantity choices, in the sprit of the theory of revealed preference by Paul Samuelson (see e.g. Afriat, 1972; and Varian, 1984; for insightful treatments of the revealed preference approach in the production setting). Still, the Malmquist TFP index only gives an inaccurate approximation in the important case where prices and/or technologies change over time, as pointed out by Balk (1993).

In this paper, we develop a modification of the Malmquist TFP index to eliminate the approximation error in case of price and/or technology change. Section 2 formally introduces the Malmquist and the Fisher ideal TFP indexes, and explores their relationship in more detail. In Section 3 we use these insights to better approximate the Fisher index by developing a modified Malmquist TFP index approach. An application to the aggregate production data of 14 OECD countries illustrates our approach in Section 5. Section 6 concludes.

2. THE FISHER IDEAL INDEXES AND THE MALMQUIST TFP INDEXES

In the elementary case where only a single output y is produced by employing a single input x, measuring the productivity change is trivially simple using the elementary productivity index

(1) $P(y^{0,1}, x^{0,1}) \equiv \frac{y^1 / y^0}{x^1 / x^0}$

where the superscripts 0 and 1 refer to the base and the target period of the index respectively. The major challenge of productivity measurement is the fact that almost all production activities involve more than just a single input and output. A simple and frequently employed solution is to look at the partial productivity indexes such as labor productivity (i.e. output per labor input). Unfortunately, such partial indexes offer an incomplete picture of productivity, since all the interesting tradeoffs between different factors are ignored. For example, high labor productivity statistics can be a signal of genuinely high level of technical efficiency, but it might equally well reflect low capital productivity.

A more ambitious strategy is to aggregate the various inputs and outputs in one way or another to construct a more comprehensive TFP index. The conventional index theory typically uses the prices (or cost/revenue) shares as the weights of quantity indexes. Of course, the prices often change from one period to another, so we inevitably face the question of which set of prices provide the most appropriate weights (consider e.g. difference between the classic Paashe and Laspayers indexes). Fisher (1922) solved this question in an ingenious way by computing two quantity indices, one using the weights of the base period and another using the weights of the target period, and taking the geometric average of the two. Specifically, denoting the output vector of period $t = 0,1$ by $y^t \in \Re^s_+$ and the associated price vector by p^t, the *Fisher ideal output quantity index* is defined as

(2) $F_o(p^{0,1}, y^{0,1}) \equiv \prod_{j=0}^{1} \prod_{t=0}^{1} \left(p^j y^t \right)^{t-1/2}$.

Similarly, denoting the input vector of period t by $x^t \in \Re^m_+$ and the associated factor prices by w^t, the *Fisher ideal input quantity index* is defined as

(3) $F_i(w^{0,1}, x^{0,1}) \equiv \prod_{j=0}^{1} \prod_{t=0}^{1} \left(w^j x^t \right)^{t-1/2}$.

Using the Fisher ideal indexes to aggregate both inputs and outputs, the *Fisher ideal TFP index* is obtained simply as the ratio of the aggregated output to the aggregated input, i.e.

(4) $TFP_F(p^{0,1}, w^{0,1}, y^{0,1}, x^{0,1}) \equiv F_o(p^{0,1}, y^{0,1}) / F_i(w^{0,1}, x^{0,1}) = \prod_{j=0}^{1} \prod_{t=0}^{1} \left(\frac{p^j y^t}{w^j x^t} \right)^{t-1/2}$.

Unfortunately, the Fisher ideal index cannot be computed if the economically relevant prices (or cost/revenue shares) are not known. As discussed in the Introduction, obtaining reliable price data is problematic in most research situations. In these circumstances, the frequently employed alternative is the Malmquist TFP index (introduced by Caves, Christensen, and Diewert, 1982a,b), which does not require any price data whatsoever.

Let the production technology be characterized by the closed and non-empty *production set*

(5) $T^t \equiv \left\{ (y,x) \in \Re_+^{s+m} \,\middle|\, x \text{ can produce } y \text{ at period } t \right\}, \quad t = 0,1$.

We define the *Malmquist TFP index* as

(6) $M(y^{0,1}, x^{0,1}) \equiv \prod_{j=0}^{1} \prod_{t=0}^{1} \left(D^j(y^t, x^t) \right)^{t-\frac{1}{2}}$,

where

(7) $D^t(y,x) \equiv Inf \left\{ \theta \,\middle|\, (y/\theta, x) \in cmc(T^t) \right\}, \; t = 0,1$

is the *Shephard output distance function*.

The distance function is conventionally defined relative to the production set T^t, which is typically assumed to exhibit monotonicity, convexity and constant returns-to-scale (CRS).[1] By contrast, we define it relative to the *convex monotone conical hull,* $cmc(T^t)$, i.e. the smallest monotone convex cone that contains T^t. This departure allows us to alleviate the standard (but nonetheless rather restrictive) assumptions that the true technology exhibits monotonicity, convexity and CRS. We need the 'benchmark technology' $cmc(T^t)$ (which exhibits these properties by definition) to establish duality between the distance measure and the economic objective function (see below). However, the true technology may be non-monotone, non-convex, and may exhibit variable returns-to-scale (VRS).

Observe that the output distance function (6) is the dual of the following equivalent 'primal' formulation:

(8) $D^t(y,x) = \underset{(\rho,\omega)\in \Re_+^{s+m}}{Sup} \left\{ \frac{\rho y}{\omega x} \,\middle|\, \frac{\rho y'}{\omega x'} \le 1 \;\; \forall (y', x') \in T^t \right\}, \quad t = 0,1$.

In contrast to the dual formulation (7), the primal formulation (8) does not require the monotone convex conical hull (although it can be used without harm). This observation is the key to understanding why Theorem 1 does not require the true technology to exhibit monotonicity, convexity and CRS.

Heuristically, the output distance function (8) can be interpreted as the ‘return-to-the-dollar’,[2] at the ‘most favorable’ prices (as represented by the weighting vector (ρ,ω), subject to a normalizing condition that no feasible input-output vectors yields the return-to-the-dollar higher than unity at those prices. We will henceforth refer to the optimal weights (ρ,ω) of (8) as the ‘*shadow prices*’ of (*y*,*x*) with respect to technology T^t. By the supporting hyperplane theorem, there exists a vector of shadow prices for any arbitrary input-output vector, but clearly, that vector need not be unique. The set of shadow price vectors is henceforth denoted by

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[1] In empirical applications, the ‘true’ technology T^t is typically unknown and must be approximated using some empirical benchmark technology. In practice, Malmquist indices are often computed using the convex monotone conical hull of the observed data points as the benchmark technology (see e.g. Färe and Grosskopf, 1996, pp. 61-62).

[2] The notion of return-to-the-dollar was introduced by Georgescu-Roegen (1951). It equals one plus the ‘profit margin’ (Jorgenson and Griliches, 1972).

(9) $$V^t(y,x) \equiv \left\{ (\rho,\omega) \in \Re_+^{s+m} \middle| \frac{\rho y}{\omega x} = D^t(y,x); \frac{\rho y'}{\omega x'} \le 1 \quad \forall (y',x') \in T^t \right\}, \quad t = 0,1 .$$

In the spirit of the theory of revealed preference, the observed allocation of inputs and outputs can indirectly reveal the economic prices underlying the production decision. In line with the definition of productivity, we assume the production vectors are set to maximize the return-to-the-dollar, and define the concept of allocative efficiency as:

Definition ('Allocative Efficiency')**:** *Production vector* (y,x) *is allocatively efficient with respect to technology* T^t *and prices* (p^t,w^t) *iff* $(p^t,w^t) \in V^t(y,x)$.

Note that allocative efficiency is obviously a necessary condition for maximization of the return-to-the-dollar. However, it is not a sufficient condition. For example, allocatively efficiency allows for technical inefficiency in the sense of Farrell (1957) (i.e. production in the interior of the production possibility set).

We are now equipped to prove the following equivalence between the Malmquist and the Fisher ideal indexes:

THEOREM 1 ('The Equivalence Theorem'): *The following conditions are equivalent:*
1) Production vectors (y^0,x^0) *and* (y^1,x^1) *are allocatively efficient with respect to prices* (p^0,w^0) *and technology* T^0*, and with respect to prices* (p^1,w^1) *and technology* T^1.
2) The Malmquist TFP index $M(y^{0,1},x^{0,1})$ *and the Fisher ideal index* $TFP_F(p^{0,1},w^{0,1},y^{0,1},x^{0,1})$ *are equivalent.*

Proof 1: If (y^0,x^0) and (y^1,x^1) are allocatively efficient with respect to both prices (p^0,w^0), (p^1,w^1) and technologies T^0, T^1, then (by definition) we find

(ii) $$\frac{p^j y^t}{w^j x^t} = D^j(y^t,x^t) \ \forall j,t \in \{0,1\} .$$

Substituting the distance functions in (6) by the revenue to cost ratios (ii) gives (4).

Q.E.D.

In the earlier literature, Diewert (1992) pointed out the equivalence between the Fisher ideal index and the Malmquist TFP index in case the distance function has a particular flexible function form. At a more general level, Färe and Grosskopf (1992) derived an analogous equivalence result to that of Theorem 1 by resorting to the following set of assumptions: 1) the production technology exhibits monotonicity, convexity, and CRS, 2) input-output vectors are allocatively efficient in terms of profit maximization.

Interestingly, Theorem 1 does not require monotonicity, convexity and/or CRS. These properties do play a crucial role; the mathematics of duality imply that the 'dual' distance measure (7) uses the 'benchmark technology' rather than the true production set.

Still, it is important to stress that the true technology need not exhibit these properties. This result partly depends on assuming return-to-the-dollar maximization rather than profit maximization; the production technology must truly exhibit CRS under profit maximization, i.e. it does not suffice to use a benchmark technology that exhibits CRS.[3] However, the objectives of profit maximization and return-to-the-dollar maximization are in fact equivalent if the technology exhibits CRS. In addition, convexity and monotonicity are not required even in case of profit maximization (although the 'dual' distance measure will use the convex monotone hull rather than the true technology as a benchmark).[4] Brief, CRS is not required in case of return-to-the dollar maximization, and monotonicity and convexity are not required in both cases. Hence, the above theorem is a true generalization of the equivalence result by Färe and Grosskopf.

As noted by Balk (1993), it seems reasonable to assume that the input-output vector of any period is allocatively efficient with respect to the technology and prices of that same period, i.e.

(10) $(p^t, w^t) \in V^t(y^t, x^t), \quad t = 0,1$

However, the Equivalence Theorem also necessitates the input-output quantities of the base (target) period to be efficiently allocated with respect to the technology and prices of the target (base) period, i.e.

(11) $(p^t, w^t) \in V^t(y^j, x^j), \quad t = 0,1;\ j = 1 - t$.

Balk (1993, pp. 680) has convincingly argued that this condition is "completely arbitrary". For instance, if the prices and/or the technology changes from the base period to the target period, then the optimal vector of the base (target) period is generally not allocatively efficient with respect to the prices and technology of the target (base) period. In that case, the Equivalent Theorem does not apply, and the Malmquist TFP index generally gives an inaccurate approximation for the ideal index.

3. A Modified Malmquist Index Approach

Resorting to some general properties of the distance functions, Balk (1993) showed that the condition of allocative efficiency suffices for using the Malmquist and Fisher indexes as reasonable first-order approximations of each other, even under price or technology changes. However, if we are interested of finding the best possible approximation for the Fisher ideal index, then we can use the above insights to derive a better approximation than the conventional Malmquist TFP index. Specifically, we focus on an exact approximation in case of unique shadow prices, and on an interval approximation in case of multiple optimal price solutions.

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[3] Note that the benchmark technology generally needs to exhibit CRS. If the benchmark set exhibits VRS, the Malmquist index need not contain the elementary productivity index (1) as its special case. Moreover, under VRS the input-based and the output-based distance functions can yield different values of the Malmquist index. Under CRS, the input and output distance functions are reciprocals to each other (Färe and Lovell, 1978), and hence the orientation of the (primal) distance function formulation does not matter for the Malmquist index.

[4] To rule out some anomalies, we need the benchmark technology to be monotonous. Specifically, in case of a non-monotonous benchmark technology, the Malmquist index can decrease even if all outputs (inputs) increase (decrease); and conversely, the Malmquist index can increase even though all outputs (inputs) decrease (increase).

To gain some additional insight, consider first a situation where the relevant (relative) shadow price vectors are unique. As explained in the previous section, the problem of the Malmquist TFP index is that it uses shadow prices of the base (target) period quantities with respect to the target (base) period technology. The approximation obviously improves if we replace these (generally) meaningless shadow prices with the meaningful shadow prices with respect to the base (target) period technology. This observation motivates us to define a tentative modification of the Malmquist TFP index as

(12) $$\tilde{M}(y^{0,1}, x^{0,1}) \equiv \prod_{j=0}^{1} \prod_{t=0}^{1} \left(\tilde{D}^j(y^t, x^t) \right)^{t-1/2},$$

where

(13) $$\tilde{D}^j(y^t, x^t) \equiv \sup_{(\rho,\omega)\in V^j(y^j, x^j)} \left(\frac{\rho y^t}{\omega x^t} \right), \quad t, j \in \{0,1\}$$

is the distance relative to the 'shadow-price augmented' benchmark set.[5] Observe the modified distance functions (13) are equivalent to the standard distance functions (8) if $t = j$; differences occur for $t \neq j$.

Interestingly, if the relative shadow prices are unique, the assumption of allocative efficiency suffices to recover the exact value of the Fisher ideal TFP index by using the modified version of the Malmquist TFP index.

THEOREM 2 ('The 2nd Equivalence Theorem')*: The following conditions are equivalent if the shadow price vectors determine unique relative prices:*

1) Production vector (y^t, x^t), $t \in \{0,1\}$ *is allocatively efficient with respect to prices* $\left(p^t, w^t\right)$ *and technology* T^t.

2) The modified Malmquist TFP index $\tilde{M}(y^{0,1}, x^{0,1})$ *and the Fisher ideal index* $TFP_F(p^{0,1}, w^{0,1}, y^{0,1}, x^{0,1})$ *are equivalent.*

Proof 2: If (y^0, x^0) is allocatively efficient with respect to prices $\left(p^0, w^0\right)$ and technology T^0, we find

(i) $$\frac{p^0 y^t}{w^0 x^t} = \tilde{D}^0(y^t, x^t), \quad t = 0,1.$$

Similarly, if (y^1, x^1) is allocatively efficient with respect to both prices $\left(p^1, w^1\right)$ and technology T^1, then

(ii) $$\frac{p^1 y^t}{w^1 x^t} = \tilde{D}^1(y^t, x^t), \quad t = 0,1.$$

[5] Using terminology of Kuosmanen and Post (2001).

Substituting the distance functions in (12) by the revenue to cost ratios (i) and (ii) gives (4). *Q.E.D.*

Figure 1 illustrates the difference to the conventional Malmquist TFP index for output based distance functions relative to a two-output technology. The two dots labeled Y^0 and Y^1 represent the output bundles of the base year 0 and the target year 1 respectively. The curves represent the frontiers of the production sets T^0 and T^1 respectively. The dotted lines illustrate the paths of equiproportionate scaling of Y^0 and Y^1 respectively, while the solid lines represent the iso-revenue surfaces at the shadow prices of Y^0 and Y^1.

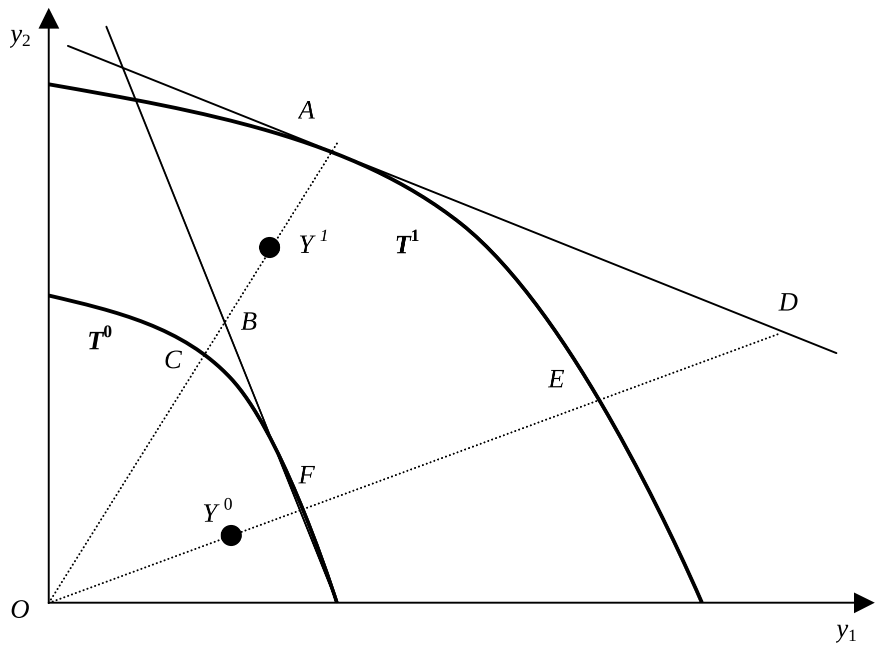

Figure 1: The conventional Malmquist index compares Y^0 to the points *F* and *E* on the technology frontiers, and Y^1 to *C* and *A*. By contrast, the modified Malmquist index compares Y^0 to the points *F* and *D* on the iso-revenue surfaces, and similarly Y^1 to *B* and *A*.

The conventional Malmquist TFP index compares Y^0 to the points *F* and *E* on the frontiers, and similarly, Y^1 to *C* and *A*; the Malmquist TFP index is given by

$$(14) \qquad M(y^{0,1}, x^{0,1}) = \left(\frac{OY^1/OA}{OY^0/OE} \frac{OY^1/OC}{OY^0/OF} \right)^{1/2} .$$

By contrast, the modified Malmquist TFP index compares Y^0 to the points *F* and *D* on the iso-revenue surface, and similarly, Y^1 to *B* and *A*, i.e. the modified Malmquist TFP index is given by

(15) $$\tilde{M}(y^{0,1}, x^{0,1}) = \left(\frac{OY^1/OA}{OY^0/OD} \frac{OY^1/OB}{OY^0/OF} \right)^{1/2}.$$

If the shadow prices correctly represent the economic prices, i.e. the condition of allocative efficiency holds, then (by Theorem 2) the modified Malmquist TFP index equals the Fisher ideal index. In that case, the conventional Malmquist TFP index is associated with the approximation error according to the formula

(16) $$\tilde{M}(y^{0,1}, x^{0,1}) = \left(\frac{OC/OB}{OE/OD} \right)^{1/2} M(y^{0,1}, x^{0,1}).$$

Balk's argument of the reasonable approximation ability of the standard Malmquist TFP index is based on the fact that both the nominator (*OC/OB*) and the denominator (*OE/OD*) of the error term are always less than or equal to unity. Still, eliminating this source of error can yield a considerable improvement, especially when there are substantial price changes leading to allocative shifts, and/or biased technology change, i.e. when (*OC/OB*) is substantially different from (*OE/OD*).

However, as noted already in the previous section, the shadow prices need not be unique. In many empirical studies, the true production set is approximated using the activity analysis (or Data Envelopment Analysis, DEA) approach. In that approach, the empirical production set has a piece-wise linear frontier that generally involves multiple shadow price vectors for the technically efficient production vectors. Unfortunately, when the shadow prices are non-unique, the assumption of allocative efficiency does not suffice for recovering the 'true' economic prices from the quantity data.

We propose to extend our tentative modification of the Malmquist TFP index in (12) to the cases with non-unique shadow prices by deriving an upper bound and a lower bound for the Fisher ideal index, derived from 'most favorable' and 'least favorable' prices. The modified distance function (13) represents the 'most favorable' prices. Similarly, we represent the 'least favorable' prices by the following distance measure:

(17) $$\tilde{E}^t(y, x) \equiv \inf_{(\rho,\omega)\in V^t(y,x)} \left(\frac{\rho y}{\omega x} \right), \quad t = 0,1.$$

Using these distance functions, we define the following upper bound for the Fisher ideal index:

(18) $$U_F(y^{0,1}, x^{0,1}) \equiv \left(\frac{D^1(y^1, x^1)}{D^0(y^0, x^0)} \frac{\tilde{D}^0(y^1, x^1)}{\tilde{E}^1(y^0, x^0)} \right)^{1/2}.$$

Similarly, we define the following lower bound for the Fisher index:

(19) $$L_F(y^{0,1}, x^{0,1}) \equiv \left(\frac{D^1(y^1, x^1)}{D^0(y^0, x^0)} \frac{\tilde{E}^t(y^1, x^1)}{\tilde{D}^1(y^0, x^0)} \right)^{1/2}.$$

THEOREM 3 ('THE INTERVAL THEOREM'): *The following conditions are equivalent:*

1) Production vector (y^t, x^t), $t = 0,1$ *is allocatively efficient with respect to prices* $\left(p^t, w^t\right)$ *and technology* T^t.

2) The Fisher index $TFP_F(p^{0,1}, w^{0,1}, y^{0,1}, x^{0,1})$ *is contained within the interval* $\left[L_F(y^{0,1}, x^{0,1}), U_F(y^{0,1}, x^{0,1})\right]$.

Proof 3: 1) $\Rightarrow$ 2). By definition, allocative efficiency guarantees that the economic prices are contained in the sets of shadow prices, i.e. $(p^t, w^t) \in V^t(y^t, x^t),\ \ t = 0,1$. Under allocative efficiency, the upper and the lower bounds satisfy following inequalities by construction:

$$L_F(y^{0,1}, x^{0,1}) \le TFP_F(p^{0,1}, w^{0,1}, y^{0,1}, x^{0,1}) \le U_F(y^{0,1}, x^{0,1}).$$

2) $\Rightarrow$ 1). $TFP_F(p^{0,1}, w^{0,1}, y^{0,1}, x^{0,1}) \in \left[L_F(y^{0,1}, x^{0,1}), U_F(y^{0,1}, x^{0,1})\right]$ immediately implies $(p^t, w^t) \in V^t(y^t, x^t),\ \ t = 0,1$, i.e. the allocative efficiency. *Q.E.D.*

The interval approximation of the Fisher index already allows for logical inference. For example if $U_F(x^{0,1}, y^{0,1}) \le 1$, then productivity decline must have occurred. Similarly, $L_F(x^{0,1}, y^{0,1}) \ge 1$ implies productivity improvement. In general, the smaller the set of shadow prices, the more narrow the interval. Moreover, the interval tends to widen when allocative shifts and technology change are considerable, i.e. in those cases where the conventional Malmquist TFP index tends to be most inaccurate.

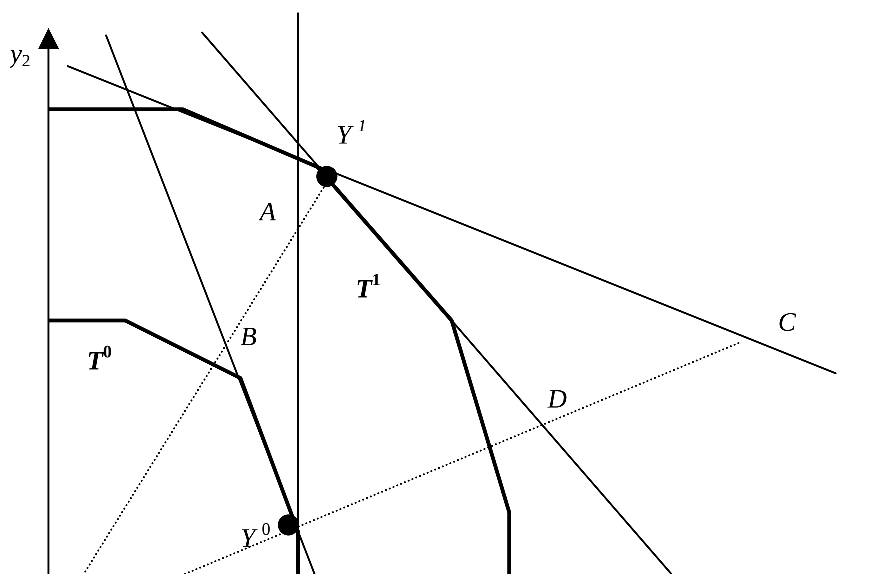

Figure 2: The case of non-unique shadow prices. The lower bound is obtained by comparing Y^0 to the point *D* and Y^1 to *A* associated with the 'least favorable' shadow prices. The upper bound compares Y^0 to *C* and Y^1 to *B* in light of the 'most favorable' shadow prices.

Figure 2 illustrates the upper and lower bounds. Again, the two dots labeled Y^0 and Y^1 represent the output bundles of the base year and the target year respectively. The solid piece-wise linear curves represent the frontiers of the production sets T^0 and T^1 respectively. Note that in this example, the evaluated unit constitutes an extreme point of the technology set in both periods. The broken lines illustrate the paths of equiproportionate scaling of Y^0 and Y^1 respectively. Finally the two solid lines associated with the both observations represent the iso-revenue surfaces at the extreme shadow prices of Y^0 and Y^1.

The minimum value of the modified Malmquist TFP index is obtained by comparing Y^0 to the point *D* on the iso-revenue surface and Y^1 to *A*, i.e.

(20) $$L_F(y^{0,1}, x^{0,1}) = \left(\frac{OY^1 / OA}{OY^0 / OD} \right)^{1/2} .$$

Clearly, this index exceeds the unity, which signals productivity improvement. The maximum value of the modified Malmquist TFP index is obtained by comparing Y^0 to the point *C* on the iso-revenue surface and Y^1 to *B*, i.e.

(21) $$L_F(y^{0,1}, x^{0,1}) = \left(\frac{OY^1 / OB}{OY^0 / OC} \right)^{1/2} .$$

Finally, if one prefers to consider a point estimate instead of the interval, an obvious alternative is to take the geometric average of the upper and the lower bounds (in the spirit of Fisher), i.e.

(22) $$\hat{M}(y^{0,1}, x^{0,1}) \equiv \left(U_F(y^{0,1}, x^{0,1}) L_F(y^{0,1}, x^{0,1}) \right)^{1/2} .$$

Again, in case the shadow prices are unique, the modified Malmquist TFP indexes (12) and (22) coincide.

4. EXAMPLE APPLICATION: TFP GROWTH IN OECD COUNTRIES

To test the modified Malmquist TFP index approach outlined in the previous sections, we undertook an application to aggregate production data of OECD countries, in the spirit of Färe, Grosskopf, Norris, and Zhang (1994). Like Färe et al. (1994), we measured aggregate output by Gross Domestic Product (GDP: measured in Mill. U.S. dollars at 1990 prices and purchasing power parity), and considered two inputs: Labor (in Thousands of employees) and Capital (Gross Capital Stock; in Mill. U.S. dollars at 1990 prices and purchasing power parity). Cross-sectional data of years 1970, 1975, 1980, 1985, 1990, and 1994 were obtained from Research Institute for Finnish Economy (ETLA) for 14 countries: Australia (AUS), Belgium (BEL), Canada (CAN), Denmark (DEN), Finland (FIN), France (FRA), Great Britain (GBR), Italy (ITA), Japan (JPN), Netherlands (NLD), Norway (NOR), Sweden (SWE), United States (USA), and West Germany (WGR). Table 1 summarizes the data set.

Table 1: Summary statistics of the output and input variables

	GDP			Labor			Capital		
	Mean	St. Dev.	Growth*	Mean	St. Dev.	Growth*	Mean	St. Dev.	Growth*
AUS	228109	62561	0,16	5740	796	0,07	902801	275414	0,19
BEL	132311	26986	0,12	3022	74	0,00	493663	134296	0,17
CAN	365282	95909	0,16	9842	1909	0,11	1623141	541238	0,20
DEN	63803	13032	0,11	2093	177	0,04	369467	73453	0,12
FIN	56515	13601	0,13	1845	157	0,00	336694	100584	0,18
FRA	775771	173145	0,13	18360	1129	0,03	2406982	620520	0,16
GBR	702660	128081	0,10	22670	702	- 0,01	2390752	537426	0,13
ITA	744208	171158	0,13	15125	761	0,02	2867872	785554	0,16
JPN	1719163	587729	0,21	45628	7247	0,09	4864027	2635125	0,42
NLD	189767	40869	0,13	4302	265	0,03	696085	148919	0,13
NOR	57215	16347	0,18	1681	218	0,07	284115	94309	0,21
SWE	112546	19381	0,09	3833	261	0,01	509968	121769	0,14
USA	4694372	1140329	0,13	90105	15083	0,09	17235742	4382503	0,15
WGR	932372	204628	0,12	23802	1467	0,03	3596828	932064	0,16
Entire Sample	769578	1231081	0,14	17718	23861	0,06	2755581	4465362	0,18

* '*Growth*' = Average growth rate of the variable from the base period to the target period

We used a CRS Cobb-Douglas production function fitted to the data set by the Corrected Ordinary Least-Squares (COLS) technique (Aigner and Chu, 1968; Richmond, 1974) as an empirical production technology. The potential specification error associated with specifying a parametric form is a disadvantage of this approach. However, the advantages of this approach (relative to non-parametric approaches) for the present application (which involves relatively small cross-sectional samples) include a relatively high robustness with respect to sampling variation, and the existence of unique and positive shadow prices. These considerations were confirmed by the outcomes of the application of the nonparametric Data Envelopment Analysis (DEA) approach (see e.g. Färe et al., 1994; Färe and Grosskopf, 1996); the DEA shadow prices for many observations equaled zero, or were non-unique[6]. For these reasons, we found the COLS approach preferable in the present application.

Table 2 summarizes the parameter estimates of the COLS regression.

Table 2: Regression results (standard errors in parenthesis)

	1970	1975	1980	1985	1990	1994
R^2	0,806	0,817	0,820	0,822	0,827	0,862
Coefficients:						
Constant	12.354	9.368	7.831	5.904	9.609	13.087

[6] Still, the distance measures estimated by the COLS and the DEA techniques were found to be highly correlated (the correlation coefficient equaled 0.958). In our interpretation, this supports the parametric specification of the COLS frontier.

Labor	0.739	0.682	0.648	0.585	0.671	0.721
	(0.0017)	(0.0016)	(0.0018)	(0.0016)	(0.0017)	(0.0013)
Capital	0.261	0.318	0.352	0.415	0.329	0.279
	(0.0016)	(0.0018)	(0.0019)	(0.0018)	(0.0018)	(0.0014)

Figure 3 displays the observations and the COLS frontiers for 1990 and 1994. Notice the biased technology change driven solely by the capital inputs. The figure also illustrates how countries almost invariably substituted labor by capital. Similar biased frontier shifts as well as substantial increases in the capital per labor were clearly identifiable in each year under study. In addition, it seems obvious that the labor compensations relative to the capital rents have increased substantially over the sample period. These considerations immediately call into question the assumption that the 1990 (1994) observations are allocatively efficient with respect to the 1994 (1990) technology, and plead for correcting the Malmquist TFP index.

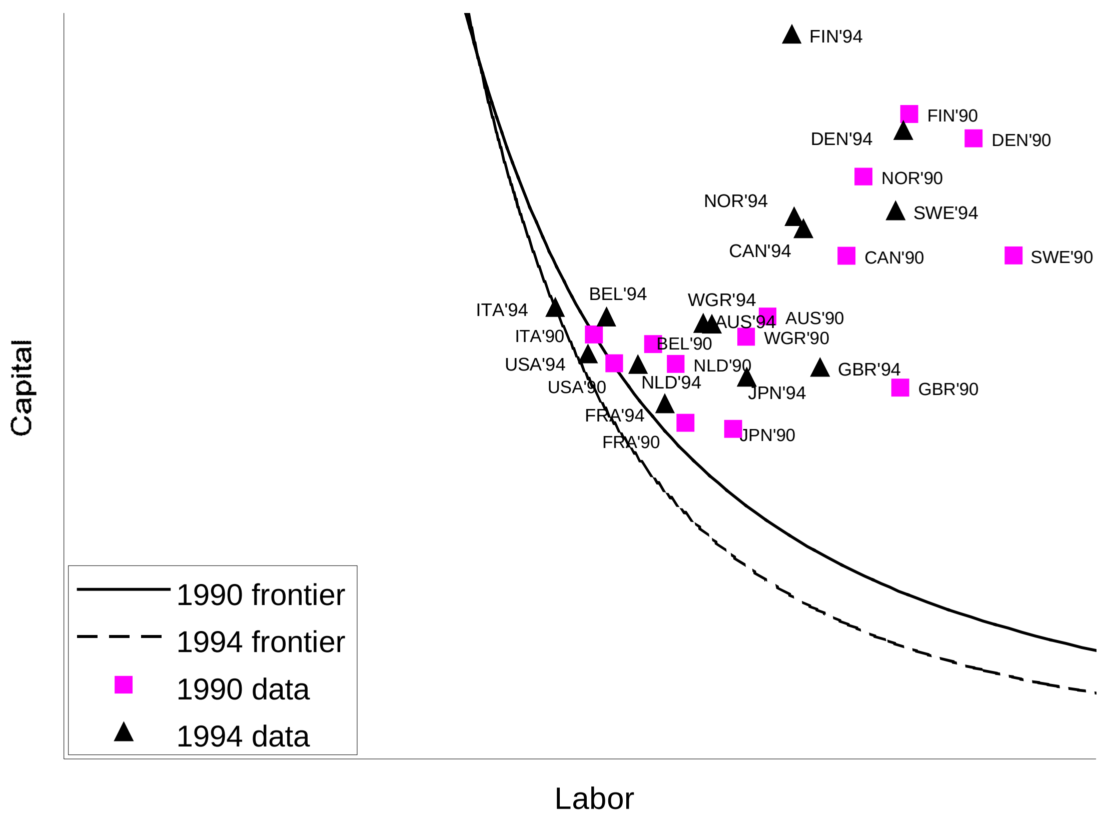

Figure 3: Estimated input isoquants for 1990 and 1994

For comparison, we computed both the conventional and the modified Malmquist indices relative to the empirical Cobb-Douglas production frontier. Table 3 summarizes the average results by country. The two indexes were found to be quite highly correlated as expected (the correlation coefficient equaled 0.82). Still, the standard Malmquist TFP index yielded invariably higher values than the modified counterpart, and hence systematically overestimated TFP growth. In fact, there were 2 cases (Japan, 1970-75; Canada, 1980-85) where the alternative indexes lead to a qualitatively opposite conclusion; i.e. the modified index suggested productivity decline, while the standard index indicated growth. On the average, the absolute values of differences varied between 0.03 – 0.06 index points. The largest difference occurred for Japan: The difference between the indexes was greater than 10 percent in 2 cases. Moreover, in total of 22 cases the difference of the two indexes exceeded 5 percent, and in 65 cases out of 70 cases the difference was greater than or equal to 1 percent. The average difference between the indexes was found to be 4.2 percent. It is worth to recall that any difference immediately constitutes an improvement for the purpose of approximating the ideal index. Figure 1 further illustrates the distribution of the differences.

Table 3: Summary of the Standard vs. Modified Malmquist indexes by Country

	Geometric average:		**Difference:**		
	Standard	Modified	Minimum	Mean	Maximum
AUS	1,0819	1,0431	0,0123	0,0356	0,0545
BEL	1,1234	1,0657	0,0333	0,0557	0,0721
CAN	1,0409	1,0143	0,0162	0,0282	0,0428
DEN	1,0698	1,0456	0,0074	0,0254	0,0414
FIN	1,1225	1,0687	0,0484	0,0612	0,0908
FRA	1,1100	1,0655	0,0262	0,0411	0,0559
GBR	1,1089	1,0618	0,0323	0,0481	0,0733
ITA	1,1131	1,0655	0,0380	0,0467	0,0652
JPN	1,1370	1,0336	0,0365	0,0900	0,1443
NLD	1,1033	1,0672	0,0017	0,0334	0,0595
NOR	1,0995	1,0476	0,0064	0,0428	0,0579
SWE	1,0668	1,0327	0,0304	0,0419	0,0732
USA	1,0406	1,0230	0,0048	0,0187	0,0343
WGR	1,0966	1,0525	0,0173	0,0413	0,0629

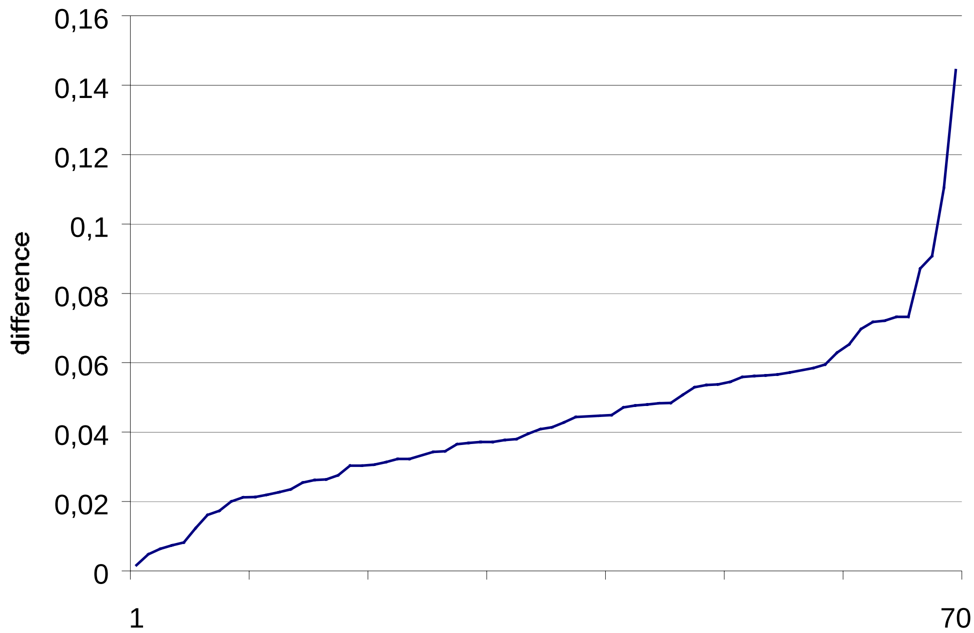

Figure 4: Distribution of the difference of the Standard and the Modified Malmquist index. The horizontal axis presents the observations sorted in ascending order (N=70).

The primary objective of this application was to see how well the conventional Malmquist TFP index approximates our modification in a typical application. The empirical results support our conjecture that the approximation error of the standard Malmquist TFP index tends to increase when the technology or prices change rapidly. It is worth to note that we used aggregate data with cross-sections from representative years. In a monthly or quarterly data the technology and price changes would tend be smaller, and the standard Malmquist TFP index could yield a more accurate approximation. On the other hand, we might expect much more dramatic allocative shifts and technical progress to occur in the firm or shop-floor level productivity analyses. Therefore, we believe the present application is representative of many research situations, and that the proposed modification can make a substantial difference in real-life applications.

7. Conclusions

In this paper, we have revisited the conditions for equivalence between the Malmquist TFP index and Fisher ideal TFP index. This question is of considerable interest due to the problem of incomplete price or value information, which frequently occurs in practical applications; in contrast to the Fisher ideal index, the Malmquist TFP index can be computed without any price (or share) data. It was earlier argued by Färe and Grosskopf (1992) and Balk (1993) (and elaborated above) that the Malmquist TFP index is a reasonable approximation of the Fisher ideal TFP index, provided that a relatively general assumption of allocative efficiency holds. This result enables one to approximate the intuitively and axiomatically attractive Fisher ideal TFP index even when the complete price information is not available.

However, as emphasized by Balk, an accurate approximation is obtained only by accident. In fact, approximation error can be quite substantial when allocative shifts (reflecting price changes) or biased technology changes occur. This observation motivated us to modify the standard Malmquist TFP index to minimize approximation error. Focusing solely on the meaningful shadow prices, we constructed upper and lower bounds for the Fisher ideal index. Whenever the shadow prices support unique relative prices, the exact value of the Fisher index can be recovered with full accuracy.

To illustrate the practical application of the proposed method and to demonstrate the gains achievable by it, we applied our method to aggregate production data of 14 OECD countries. The empirical results demonstrate that the proposed modified indexes can substantially improve estimation in real-life applications.

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APPENDIX

This appendix presents the Linear Programming formulations for computing the modified distance functions relative to a Data Envelopment Analysis (DEA) frontier.Let the output data of period t be denoted by the matrix $Y^t = \left(y_1^t \cdots y_n^t\right)$, t = 0,1; and use $X^t = \left(x_1^t \cdots x_n^t\right)$ for the matrix of input vectors respectively. We start by computing the standard distance functions D by solving the following two Linear Programmin (LP) problems of the standard form:

$$D^t(y_k^t, x_k^t) = \underset{\rho,\omega}{Max}\, \rho y_k^t$$
$$s.t.\;\; \omega x_k^t = 1$$
$$\rho Y^t - \omega X^t \le 0$$
$$\rho, \omega \ge 0$$
$$t = 0,1$$

We next use the shadow prices associated with the optimal solutions of the above LP problems to obtain the modified distance functions. This boils down to solving the following LP problems

$$\tilde{D}^j(y_k^t, x_k^t) = \underset{\rho,\omega}{Max}\, \rho y_k^t$$
$$s.t.\;\; \omega x_k^t = 1$$
$$\rho Y^j - \omega X^j \le 0$$
$$\rho y_k^j = \omega x_k^j D^j(y_k^j, x_k^j)$$
$$\rho, \omega \ge 0$$
$$t = 1 - j$$
$$j = 0,1$$

and

$$\tilde{E}^j(y_k^t, x_k^t) = \underset{\rho,\omega}{Min}\, \rho y_k^t$$
$$s.t.\;\; \omega x_k^t = 1$$
$$\rho Y^j - \omega X^j \le 0$$
$$\rho y_k^j = \omega x_k^j D^j(y_k^j, x_k^j)$$
$$\rho, \omega \ge 0$$
$$t = 1 - j$$
$$j = 0,1$$