Duality Theory of Non-convex Technologies

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Abstract:
Duality Theory of production imposes a number of simplifying assumptions regarding the production technology, including various maintained convexity assumptions. Emphasizing the technological information content of alternative models, this paper challenges some widely held views on the role of convexity. The role of convexity in Duality Theory is asymmetric: While convexity is of importance in recovering technology information from economic models, cost functions are concave and profit functions are convex irrespective of convexity of the underlying technology. For recovering technology information from economic models and data, we discuss two alternative approaches: recovering inexact outer-bound approximations; and enriching standard economic models by additional quantity/financial constraints. The main conclusion is that non-convexities should not stop one from applying the duality theory.

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1. Introduction

The theory of production offers a wide variety of alternative models for representing the production possibilities. Data requirement of alternative models differ: Directly production-oriented models focus on the input-output quantities, while economists often prefer to study monetary price, cost, revenue, and/or profit data. Rooted in the seminal work of Hotelling (1932) and especially Shephard (1953, 1970, and 1974), *Duality Theory* of production offers an axiomatic framework that links alternative models in a systematic, rigorous fashion.[1] Although Duality Theory is a formal axiomatic system void of empirically testable hypotheses, it is more than a mere classification system.[2] It has powerful methodological implications, which has deservedly earned paramount appreciation among applied economists. A prime example is the possibility to derive technological properties (like scale or substitution elasticities) indirectly from monetary price and profit (or cost/revenue) data, without physical quantity/volume data needed to estimate production functions or other technology representations. Another major thrust of Duality Theory comes from analysis of input demand and output supply using such famous results as Hotelling's lemma, Roy's identity, and Shephard's lemma. From practical/applied perspective, accommodating joint production (i.e., multiple-input multiple-output technologies) in traditional regression analysis using cost, revenue, and profit functions as dual representations of technology has been one major practical benefit of Duality Theory, among many others.

Generality of Duality Theory is one of its distinct strengths. The modern Duality Theory can effectively deal with multiple-input multiple-output technologies, without restricting to any particular function forms or parametric structures in its production and economic models. In some other important dimensions, however, the scope of Duality Theory is rather limited: (1) Duality Theory offers a static view on the firm. The subtle dynamics of production are almost always ignored (a notable exception is Färe, 1978). (2) The economic models of Duality Theory focus on the price-taking firm, assuming away any dependence between the volume of transaction and the price. (3) Duality Theory usually assumes the full certainty both in its economic and production models, thus ignoring the various risks and uncertainties related to the outcomes of the physical production processes as well as the market mechanisms, and the firms' willingness and ability to bear them. (4) The production models of Duality Theory always assume convexity in one form or another. To link an extensive collection of economic models and production models together in the powerful "duality diamond" (Färe and Primont, 1994, 1995), Duality Theory requires the entire production possibility set to be convex.

I personally find the fourth limitation the most 'troubling' one. For its convenience, I can live with the crude static model of the risk-neutral price-taking firm as a reasonable approximation applicable for many practical purposes. After all, the problem of data availability often serves as a natural barrier for empirical applications where this neoclassical view is grossly violated. Perhaps most importantly, we economists are generally well equipped with necessary conceptual tools and training for evaluating the market structure and its special features, which might call upon refinements to the neoclassical model, even its rejection. By

[1] In economic theory, the production duality has a direct analogy with the duality theory of consumption and demand analysis, see e.g. Diewert (1974, 1982). From mathematical point of view, most economic applications of duality theory apply the theorem by Minkowski (1911), stating that every closed convex set in $\mathbb{R}^m$ can be characterized as the intersection of its supporting half-spaces.

[2] To quote Eggleston (1958, p. 25): "Duality theory … often suggests alternative proofs of known results; it suggests new results which are "dual" to known results; it helps to clarify existing knowledge and to coordinate diverse results."

contrast, the convexity properties required by Duality Theory are abstract, purely technical conditions, which fall far beyond the scope of economic reasoning, and have proved very difficult –if not impossible- to test by rigorous statistical/econometric techniques. Almost always, we must take these convexity properties by faith.

Convexity can be viewed as a second-order curvature condition of the production frontier. In theory, the convexity postulates are sometimes justified by deriving them from a set of more elementary and intuitive axioms, like *additivity* and *divisibility* (e.g., Arrow and Hahn, 1971). This, however, does not make the case for convexity any stronger. In this perspective, the divisibility axiom (or its violation) is the key to the (non-)convexity. Not only all inputs and outputs should be divisible in infinitesimal fractions, divisibility also necessitates that downsizing the entire production process should be feasible.[3] Therefore, divisibility (and hence convexity) assumes away such economically important technological features like *economies of scale* and *economies of specialization*.[4] These are probably the two most interesting technological features for the economist, but ironically, the dual approach to production theory tends to deny their existence.

The objectives of this paper are two-fold: The first objective is a more elaborate and profound understanding of the asymmetric role of convexity in Duality Theory, and its empirical implications. Economy with maintained assumptions is a very important feature of any theory, especially the fundamental Duality Theory. The fact is, not only the sound formal results of Duality Theory matter, also its maintained assumptions guide model specification in empirical applications. Unfortunately, applications of duality theory often impose entirely redundant assumptions, just to be on the safe side. On the other hand, the failure of establishing duality is sometimes (falsely) interpreted as a problem of the non-convex technologies.[5] In my opinion, the full potential of Duality Theory - as well as its limits - in dealing with such interesting and important technological features like economies of scale deserves clarification.

The second purpose is to extend or generalize Duality Theory to deal with non-convexities endogenously within the theory. Although Duality Theory relies heavily on convexity, some isolated remarks to alleviate convexity can be found in the literature. A good starting point is the observation that the *affine* structure of the usual (neo-classical) firm objectives like cost minimization or profit maximization is the ultimate source of convexity in Duality Theory. This gives rise to two immediate routes for extensions, which also constitute the dual structure of this paper:

I) We may distinguish between the "true" production possibilities and their outer approximation. The convex technology set induced by an economic model gives an outer bound approximation, i.e., the convex hull of the underlying true technology set, even if the

[3] This point was aptly made by M.J. Farrell (1959, pp. 378 – 379) in his article "The Convexity Assumption in the Theory of Competitive Markets" (Section II entitled ' The importance of non-convexities'):
"A glance at the world about us should be enough to convince us that most commodities are to some extent indivisible and that many have large indivisibilities. Similarly, whenever one refers to "economies of scale" or of "specialization", one is pointing to concavities [=departures from convexity] in production functions. There is thus no need to argue the importance of either indivisibilities or concavities in production functions - the former are an obvious feature of the real world, and the latter have constituted a central topic in economics since the time of Adam Smith."
[4] Some fascinating economic analyses stemming from these types of violations of convexity include Yang (1994), Yang and Ng (1993), Borland and Yang (1995), Shi and Yang (1995), and Yang and Rice (1994).
[5] For example, Thrall (1999) has explicitly suggested that non-convex technologies are void of any real-world economic meaning. In my opinion, this view is unsustainable. We refer to Cherchye, Kuosmanen, and Post (2000) for the rebuttal.

true technology is non-convex. Hence, non-convexities need not prevent us from applying Duality Theory. This possibility has attracted only passing remarks in the literature of Duality Theory. For example, Diewert (1972, pp. 109-110) and Russell (1998, pp. 47) briefly mention it, but do not develop it further. To the best of my knowledge, a fully-blown rigorous analysis supported with formal proofs is missing. This could explain why the application possibilities of this argument have not been utilized up to their full potential. This paper intends to fill in this gap by presenting a systematic review of the envelopment sets induced by most standard technology representations in the general multi-input multi-output setting. Although some "new" duality relationships will be identified, I think the main novel content in this respect concerns the overall treatment, rather than any single duality theorem per se.

From methodological point of view, it is also interesting to note a direct link to two other prominent schools of thought in the production analysis literature: The *Afriat-school* (Afriat, 1972; Hanoch and Rothschild, 1972; Varian, 1984) and the *Charnes-Cooper school* (Charnes, Cooper, and Rhodes, 1978; Banker, Charnes, and Cooper, 1984) [in terminology of Førsund and Sarafoglou, 2002]. A common feature of both these research paradigms is the envelopment of the observed input-output quantity data with conical, convex, and/or monotonic hulls. That is, covering the "cloud" of observed points, which is a discrete non-convex set, with the minimal set that has a certain desirable shape. The key difference to bear in mind is that we here do not envelope the observed data but the true underlying technology itself. Therefore, the envelopment sets of this paper should be viewed as *outer* bound approximations, where as most of the current empirical literature uses the envelopment sets as *inner* bounds. Still, many of the conclusions of this paper carry directly over to these more empirical approaches if we take the discrete set of observations as the estimator of the underlying unobserved production possibility set. Therefore, I would like to see this analysis as one step towards a more unified framework of production analysis, which links some intimately related but separately developed streams of literature more closely together, and accounts for non-convexities endogenously within the theory.

II) We may substitute the simple affine economic objectives, which cannot capture non-convexities, by more complicated non-linear objectives that can represent such features. This second class of extensions directly relates to other limitations of Duality Theory listed above. In particular, Cherchye, Kuosmanen, and Post (2000) and Kuosmanen and Post (2002) have earlier pointed towards imperfect competition (i.e., price-making rather than price-taking) and risk aversion under price uncertainty as circumstances under which the firm objectives become non-linear and hence non-convexities of the technology influence the optimal production plan of the firm. In the similar vein, extensions of duality theory where non-linear constraints give rise to non-convex preferences are known in the theory of consumer choice, see e.g. Epstein (1981) and McCabe (1997). In mathematics, the classic duality techniques have been extended to non-convex cases by Balder (1978) using particular non-linear "needle-type" support functions. Balder' s approach has been recently applied to the general equilibrium theory of non-convex technologies and preferences by Joshi (1997). Finally, relating to the present production context, First, Hackman, and Passy (1993) have suggested the notion of *projective-convexity* (or shortly *P*-convexity) as tool for extending towards non-convex environments.

This paper will not follow any of the previous developments. Rather, we stick firmly to the traditional affine objectives considered by Shephard, McFadden, Diewert, and others. I will argue that non-convexities matter even in this traditional framework. The standard quantity and price constraints, which are an intimate part of the traditional Duality Theory [consider the cost function, for example], suffice for capturing (some) non-convexities. Inspired by McFadden' s

(1978) restricted profit function, we will present a general economic model, which is a complete representation of technology capable of capturing non-convexities and congestion.[6]

The organization of the subsequent sections is the following. We start in Section 2 by introducing the basic (primal) production models defined in terms of input-output quantities, and analyze the technology information content of alternative models. We then move in Section 3 to the basic (dual) economic models defined in terms of monetary variables, and derive the associated dual envelopments. Section 4 extends the scope of analysis to indirect models, which include some financial budget constraints or turnover targets. In Section 5 we introduce a novel concept of constrained profit function, and investigate its relationship to other economic models as well as its technology information content. Section 6 discusses the more practical implications of this study. As for our overall objectives, extensions of type I (identified above) are found in Sections 2 - 4, while generalizations of type II concentrate on Section 5. Of course, each and every section is there to contribute to our understanding of convexity in Duality Theory.

Finally, some technical notes: Since this treatment involves a large number of different models, most of which are standard and well-known, we choose to summarize most model definitions in form of tables to keep the discussion concise. As usual, the key results are presented in the form of mathematical theorems. Since the mathematical proofs of these theorems are more of technical rather than deeply innovative nature, we will only prove the least trivial theorems, and place the proofs in Appendix.[7]

2. Basic Production Models

The general multi-output production model describes the alternative means of transforming resources to commodities. The static theory of production conventionally uses the point-to-set topology as its analytical framework.[8] Production outcomes are thus conveniently modeled as points of an Euclidean vector space, and the production technologies as its subsets. Assuming *m* relevant factors, let $y = (y_1 \ldots y_m) \in \mathbb{R}^m$ represent the vector of *net throughput* (or shortly *netput*), where elements $y_i < 0$ are interpreted as inputs and elements $y_j > 0$ are outputs. We do not assume that subsets of inputs and outputs are exogenously given to the firm. Rather, the roles of inputs/outputs might be reversed (if the firm finds it advantageous to do so), depending on the technological possibilities and the prices.[9] For example, some 'intermediate factors' might be consumed as inputs or produced as outputs depending on the prices.[10]

[6] Congestion refers to lack of free disposability. See e.g. Cherchye, Kuosmanen, and Post (2001) for discussion.

[7] According to the Nobel Prize-winning physicist Richard Feynman (Feynman and Leighton, 1997), mathematicians label any theorem as "trivial" once a proof has been obtained, so there are hence only two types of true mathematical theorems: trivial ones, and those which have not yet been proven.

[8] Using the standard notation, $\mathbb{R}^m$ denotes the *m* dimensional Euclidean vector space. The subscripts have the following meaning: $\mathbb{R}_+ \equiv [0,\infty)$, $\mathbb{R}_{++} \equiv (0,\infty)$, and $\mathbb{R}_- \equiv (-\infty, 0]$. Moreover, $\overline{\mathbb{R}} \equiv \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the set of affinely extended real numbers.

[9] Consider e.g. the possibility of outsourcing some functions of the firm. It may be worth elaborating that the possibility to reverse the roles of input/output does not yet violate *irreversibility* implied by the Second Law of Thermodynamics: Even if we in principle allow for transforming an 'input' to an 'output', and again back to the 'input', we do not assume we achieve the same amount of input as in the beginning of the process.

[10] In words of Lau (1974, pp. 180): "After all, the decision whether to purchase or to sell a given commodity must be an economic one. Hence, one cannot arbitrarily classify certain commodities as outputs and others as inputs, except for commodities which inherently cannot be produced by any production process, such as unskilled labor, or for commodities which are prior dated. This symmetric treatment has many potential applications: analysis of

The production technology can be characterized in a number of alternative ways, for example, using the traditional production functions or the technology distance functions. Since the set representations of technology offer the most immediate platform for studying convexity, we shall adopt the production possibility set $T \subset \mathbb{R}^m$ as my default production model. This set is simply defined as the list of all feasible netput vectors, i.e.,

(1) $T \equiv \left\{ y \in \mathbb{R}^m \middle| \text{netput } y \text{ technically feasible} \right\}$.

To rule out some anomalies, the following set of maintained axioms are assumed to hold:

(A1) $T \neq \emptyset$ (non-emptyness).
(A2) $T \cap \mathbb{R}^m_+ = \left\{ \vec{0} \right\}$ (inactivity possible, no free lunch).
(A3) T is a closed set.
(A4) $\left\{ y \in T \middle| y \geq y' \right\}$ is a bounded set $\forall y' \in T$ (scarcity).

These 4 maintained axioms comprise most standard maintained axioms of the production theory (e.g. Färe and Primont, 1995, p. 27). However, we deliberately deviate from earlier treatments of Duality Theory in that we do not postulate any form of convexity or disposability of the true underlying production technology. When the limited technological information content of other models dictates a particular topological shape, we will envelope the true technology set by its smallest convex superset that has the desired shape. In other words, we draw a sharp distinction between a simplified "proxy" and the "real thing". For completeness, Table 1 formally defines the standard envelopment notions in terms of an arbitrary set $S \subset \mathbb{R}^k$.[11]

Table 1: Envelopments of set $S \subset \mathbb{R}^k$

Convex hull	$conv(S) \equiv \left\{ \sigma \in \mathbb{R}^k_+ \middle| \sigma = \alpha\sigma' + (1-\alpha)\sigma''; \sigma', \sigma'' \in S; \alpha \in [0,1] \right\}$
Monotonic hull	$mon(S) \equiv S - \mathbb{R}^k_+$
(Partial) conical hull	$coni^{[a,b)}(S) \equiv \lambda S, \lambda \in [a,b)$
Convex monotonic hull	$cm(S) \equiv conv(S) \cup mon(S)$

Other production models typically assume distinct inputs and outputs. In case the first *r* netputs happen to be non-positive, i.e., inputs denoted by $x \in \mathbb{R}^r_-$, and the remaining *s* netputs are non-negative, i.e., outputs $u \in \mathbb{R}^s_+$, $r + s = m$, we will find it convenient to partition the netput vector y as $y = (x, u)$. The distinction between inputs x and outputs u is totally immaterial for this paper, and will be invoked merely to establish connections to the earlier literature. Table 2 summarizes the production models considered in this paper. All alternative models are well established in the literature, see e.g. Färe and Primont (1995) and Chambers et al. (1998) for further details.[12]

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international trade; analysis of conglomerates; analysis of integration and mergers; and analysis of agricultural households."

[11] The notion of conical hull is here generalized in a straightforward fashion to allow for partial cones. The standard conical hull is the special case obtained as $coni^{[0,\infty)}(.)$.

[12] For consistency and analytical convenience, we here deviate from the standard notation by preserving the negative sign of the inputs.

Table 2: Alternative Production Models

Model	Formal definition
Input correspondence	$L:\mathbb{R}^s_+ \to 2^{\mathbb{R}^r_-}$, $L(u) \equiv \{x \in \mathbb{R}^r_- \mid (x,u) \in T\}$
Output correspondence	$P:\mathbb{R}^r_- \to 2^{\mathbb{R}^s_+}$, $P(x) \equiv \{u \in \mathbb{R}^s_+ \mid (x,u) \in T\}$
Input distance function	$D_i : \mathbb{R}^r_- \times \mathbb{R}^s_+ \to \overline{\mathbb{R}}_+$, $D_i(x,u) \equiv \sup_\theta \{\theta \in \mathbb{R}_+ \mid (x/\theta, u) \in T\}$
Output distance function	$D_o : \mathbb{R}^r_- \times \mathbb{R}^s_+ \to \overline{\mathbb{R}}_+$, $D_o(x,u) \equiv \inf_\theta \{\theta \in \mathbb{R}_+ \mid (x, u/\theta) \in T\}$
Directional distance function	$DD:\mathbb{R}^{2m} \to \overline{\mathbb{R}}_+$, $DD(y,g) \equiv \sup_\theta \{\theta \in \mathbb{R}_+ \mid y + \theta g \in T\}$

For notational convenience, each model is assumed to represent the same production technology [i.e., *T*], unless otherwise indicated. We will use the superscript to indicate alternative technology domains. For example, $D_i^{cm(T)} \equiv \sup_\theta \{\theta \in \mathbb{R}_+ \mid (x/\theta, u) \in cm(T)\}$.

It is immediately obvious that knowledge of *T* suffices for deriving any of the alternative production models of Table 2: After all, every model of Table 2 is defined in terms of *T* to begin with. But can we also recover *T* from the other production models? If not, what can we then say about the technology?

The *input correspondence L* is a point-to-set mapping, which relates output vector *y* to the set of input combinations that can produce it. That is, the image of *L* is a subset of $\mathbb{R}^r_-$. On the other hand, the *output correspondence* gives the set of output vectors producible by a given input *x*. It is rather well known that the input and the output correspondence are generally equivalent to *T* as production models (see e.g. Färe and Primont, 1995, p. 19). It is worth stressing here that, besides the maintained axioms and the ability to draw distinction between inputs and outputs, this equivalence does not depend on any simplifying topological assumptions.

Theorem 2.1*: If netputs can be partitioned to inputs and outputs, i.e.,* $y = (x,u), x \le \vec{0}, u \ge \vec{0}$, *then the input correspondence L and the output correspondence P are general technology representations equivalent to the production possibility set T. That is,*

$$x \in L(u) \Leftrightarrow u \in P(x) \Leftrightarrow (x,u) \in T .$$

This useful fact will be employed in the following, as in many cases it is more convenient to phrase in terms of the input and the output correspondence than using the set *T*. The next two models are prime examples.

The Shephard (1953) *input distance function* D_i indicates the maximum equiproportionate contraction factor of input vector *x* such that the given output *y* is producible. Reciprocal of the Farrell (1957) input efficiency measure, this traditional distance function plays traditionally an important role in the productivity and efficiency analysis. Due to its inherent input orientation, the input correspondence *L* is a natural dual partner for investigating the technology information contained by D_i. We can link the two models by the following theorem:

Theorem 2.2*: The technology information of the input distance function D_i is equivalent to a partial conical hull of the image of L, specifically,*

$$D_i(x,u) \geq 1 \Leftrightarrow x \in coni^{[1,\infty)}(L(u)) .$$

Proof: See Proof 2.2 in Appendix.

It is known that the input distance function can perfectly represent the production technology, provided that inputs are *weakly disposable*. Theorem 2.2 confirms this fact, since weak input disposability is equivalent to the condition $L(u) = coni^{[1,\infty)}(L(u)) \ \forall u \in \mathbb{R}_{+}^{s}$. What is new in Theorem 2.2 is the fact that the input distance function can provide us with detailed information of the technology even when weak disposability fails to hold. Although the input distance function cannot account for some extreme forms of input congestion, it adequately accounts for violations of convexity as well as free disposability.

The *output distance function* D_o (also by Shephard, 1953) indicates the maximal equiproportionate expansion of output vector *y* producible with the given input *x*. It can be viewed as a mirror image of the input distance function. By reversing the roles of inputs and outputs, Theorem 2.2 can be re-expressed in terms of outputs, to link the output distance function and the output correspondence *P*. This is formalized by the following theorem.

Theorem 2.3*: The technology information of the output distance function D_o is equivalent to a partial conical hull of the image of P, in particular,*

$$D_o(x,u) \leq 1 \Leftrightarrow u \in coni^{[0,1]}(P(x)) .$$

Proof. Directly analogous to Proof 2.2 in Appendix, and hence omitted.

Finally, we briefly consider the more general *directional distance function DD* recently proposed by Chambers, Chung, and Färe (1996, 1998), inspired by Luenberger's (1990) benefit function. The directional distance function projects the given point *y* to the boundary of *T* in an endogenously specified direction $g \in \mathbb{R}_{++}^{m}$. Chambers et al. (1996, 1998) have shown that *DD* is equally good technology representation as traditional D_i or D_o. In fact, it is even more general. The minimal conditions for recovering *T* from *DD* are given by the following theorem:

Theorem 2.4*: The information content of the directional distance function DD can be summarized by the following envelopment of the production possibility set T:*

$$\min_{g \in \mathbb{R}_{++}^{m}} DD(y,g) \geq 0 \Leftrightarrow y \in \bigcap_{g \in \mathbb{R}_{++}^{m}} \left\{ y \in \mathbb{R}^{m} \middle| y = y' - \theta g; y' \in T; \theta \geq 0 \right\} .$$

Proof. See Proof 2.4 in Appendix.

More intuitively interpreted, *DD* provides the exact representation of the technology if for every infeasible input-output vector there exists at least one non-negative direction vector which yields strictly negative value of the directional distance function. Put differently, the directional distance function cannot identify an infeasible point if it is ' masked' or surrounded by feasible input-output vectors in all positive directions. For example, the directional distance function cannot distinguish a *punctured set* $T-\{(x',y')\},(x',y')\in \mathrm{int}(T)$ from *T*. Figure 1 illustrates by a simple graphical example the envelopment induced by the directional distance function. The black area represents an arbitrary non-convex, non-monotonic set of feasible input-output combinations, while the white represents the infeasible points. The gray color indicates the infeasible points, which the directional distance function falsely identifies as feasible.

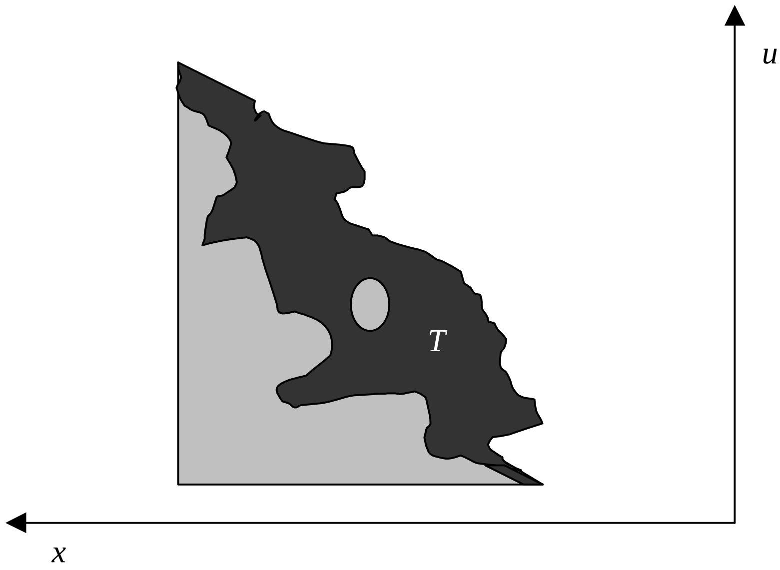

Figure 1: Example of the envelopment (the black and gray area) of an arbitrary non-convex set $T\subset\mathbb{R}^2$ (the black area) induced by the directional distance function.

We may safely interpret that, despite some limitations, the directional distance function generally contains all information of the technology that could be of economical interest. In any case, the directional distance function proves the most general technology distance function. More specifically, it is easy to show that the above-discussed envelopments are nested in the following sense

(2) $$T\subseteq\left\{y\in\mathbb{R}^m \,\middle|\, DD(y,g)\geq 0\ \forall g\in\mathbb{R}^m_{++}\right\}\subseteq \begin{matrix}\left\{(x,u)\,\middle|\, x\in coni^{+}(L(u))\right\}\\ \left\{(x,u)\,\middle|\, u\in coni^{-}(P(x))\right\}\end{matrix}.$$

As the general conclusion, all basic production models considered thus far are sufficiently general to account for violations of convexity and free disposability. Only some rather minimal disposability conditions are required for recovering production technologies form technology distance functions. Convexity does not have a role in the "duality" of the basic production models.[13]

3. Economic Models

The economic model describes the objectives and the constraints of the firm. As economists are often more comfortable with monetary price/cost/revenue/profit data than with

[13] The notion of ' duality' typically refers to technology representations that use dual variables, for example, prices and profit. Although the distance/gauge functions are sometimes called dual representations of technology, they are not dual in this vector space sense.

physical quantities, the possibility to analyze production technology indirectly through economic models lies in the heart of Duality Theory. Following the conventional approach, we confine attention to the price-taking firm that maximizes monetary gain or minimizes the loss subject to the technological constraints represented by *T*, and possibly some additional quantity constraints.

The three basic models formally defined in Table 3 will be considered in this section. Indirect models involving budget constraints and/or revenue targets will be discussed in the next section. As usual, we shall assume the firm takes the netput prices $p \equiv (p_1 \ldots p_m) \in \mathbb{R}_+^m$ as given. When inputs and outputs are separable, we partition *p* in input price vector $w \equiv (w_1 \ldots w_r) \in \mathbb{R}_+^r$ and output price vector $v \equiv (v_1 \ldots v_s) \in \mathbb{R}_+^s$.

Table 3: Alternative Economic Models [14]

Model	Formal definition
Profit function	$\Pi : \mathbb{R}^m \to \overline{\mathbb{R}}$, $\Pi(p) \equiv \sup_y \{ p \cdot y \mid y \in T \}$
Cost function	$C : \mathbb{R}_+^{r+s} \to \overline{\mathbb{R}}_+$, $C(w,u) \equiv -\sup_x \{ w \cdot x \mid x \in L(u) \}$
Revenue function	$R : \mathbb{R}_-^r \times \mathbb{R}_+^s \to \mathbb{R}_+$, $R(x,v) \equiv \max_y \{ v \cdot y \mid y \in P(x) \}$

The *profit function* (Hotelling, 1932) is the standard model of the neoclassical microeconomics. It indicates the maximum profit achievable at the given input-output prices. The key properties of the profit function include the following:

Theorem 3.1: *The profit function* Π *is non-decreasing, homogenous of degree one, convex, and continuous in prices p.*

Proofs of this famous theorem go back at least to Samuelson (1953-1954, pp. 20), and can be found in any comprehensive microeconomics textbook. What is worth noting here is that many treatments (especially the "pedagogical" ones) tend to impose a large number of redundant assumptions, including convexity of the production possibility set.[15] Yet, it is simple to verify that the profit function defined over a non-convex *T* always equals the profit function defined over the convex hull of *T*, i.e., $\Pi^{conv(T)}(p) = \Pi^T(p) \; \forall p \in \mathbb{R}_+^m$. Therefore, if profit function is convex in prices in case of the convex hull of *T*, then it must be convex also under *T*. In other words, non-convex technologies give rise to well-behaved profit functions equally well as convex ones! The axiom (A3) of closed technology is already a sufficient condition for Theorem 3.1. In this sense, the domain of the profit function is actually much more general than most textbooks suggest.

One of the prime results of Duality Theory is establishing that the profit function, which is defined solely in terms of monetary variables, offers in fact an implicit technology representations equivalent to production possibility set *T*. However, this strong equivalence

[14] A finite profit maximum need not exist for all technologies and price vectors (e.g., $T = \{(x,u) \in \mathbb{R}_+^2 \mid -x \geq u\}$ and *v* = 2, *w* = 1). Therefore, it is conventional to assign the profit function the value of infinity in those cases. Similarly for the cost function, the output target *u* may technically infeasible whatever amounts of inputs are used. The cost function equals infinity in such cases. For revenue function, however, the scarcity axiom (A4) suffices to guarantee that a finite maximum revenue exist for any given inputs and output prices.

[15] See the famous treatments of McFadden (1978, pp. 81) and Diewert (1974, pp. 136), for example.

result does indeed depend on the economically restrictive assumptions of convex, freely disposable *T*. Hence the scope of the traditional duality relationship is limited. Nevertheless, we can quite easily avoid all these restrictive assumptions, if only we are willing to settle for an inexact outer approximation of the production possibility set whenever the assumptions fail to hold. The following theorem formalizes the generalized duality relationship:

Theorem 3.2: *The technology information of the profit function* Π *is equivalent to the convex monotonic hull of the production possibility set T, that is,*

$$\left\{ y \in \mathbb{R}^m \,\middle|\, p \cdot y \leq \Pi(p) \;\; \forall p \geq \vec{0} \right\} = cm(T) .$$

Proof. See Appendix, Proof 3.2.

The profit function does not generally suffice for recovering the exact production possibility set: Based on the profit function, it is generally not possible to infer whether an arbitrary $y \in cm(T)$ is truly technically feasible or not. The standard theory simply assumes feasibility. For generality, we here refrain from such assumptions. By contrast, we stress that an outer 'envelopment' - the convex monotonic hull of *T* - can always be recovered, even when the underlying technology is non-convex. And conversely, the knowledge of the convex monotonic hull of *T* suffices for recovering the profit function associated with *T*, even if *T* itself is not known. In some applications, this outer approximation may give sufficient technology information even though the possible economies of scale and of specialization as well as congestion effects cannot be detected. To get a more detailed picture on these issues, one should either use a more detailed economic model or investigate the production technology directly in terms of a suitable production model.

Technical note: By Theorem 3.2, the graph of profit function $graph(\Pi) \equiv \left\{ (p, \Pi(p)) \in \mathbb{R}^{m+1} \,\middle|\, p \geq 0 \right\}$ is the *polar set* transformation of $cm(T)$. Alternatively, Π can be viewed as the support function of $cm(T)$.

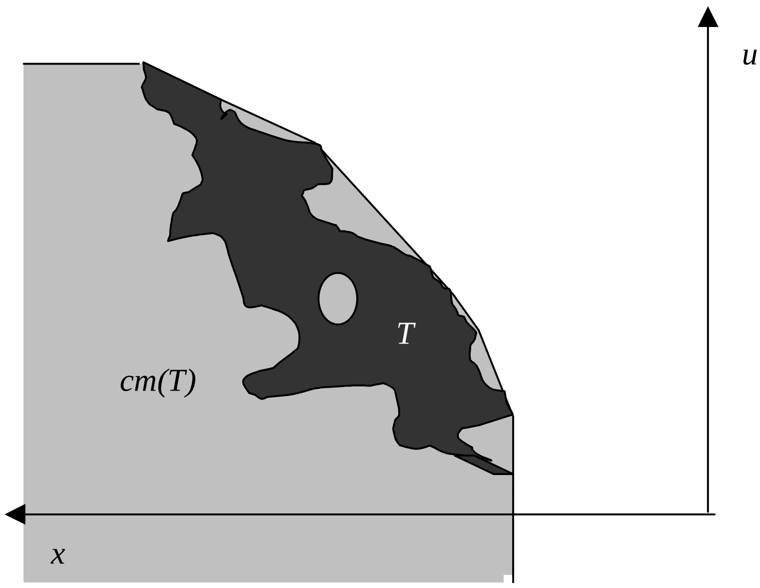

Figure 2: Example of the envelopment of an arbitrary non-convex set $T \subset \mathbb{R}^2$ ***(the black area) by convex monotonic hull (the black and gray area).***

Figure 2 graphically illustrates the convex monotonic hull of a non-convex set, following the example of Figure 1. The boundary of *cm*(*T*) represents netput combinations which a profit-maximizing firm may choose under perfect free competition without constraints. The gray area represents the netput combinations included in the envelopment set, which are technically infeasible. Although the gray area only gives a very rough approximation of *T* in this case, it may be better than nothing.

The profit function gives a stylized description of a price-taking firm, which considers all inputs and outputs as decision variables to be optimized. In practice, most firms must operate with a smaller number of decision variables. Many industries face some temporal or permanent limits in the input factors, e.g., capacity constraints or shortage of skilled labor in the short-run, and scarcity of natural resources and land area in the long-run. Similarly, the firm may have some prior commitments to certain delivery conditions for outputs. In addition, both inputs and outputs may be subject to quantity rationing or regulation. For public sector non-for-profit firms, the input resources or the output targets are often exogenously determined. These types of quantity restrictions are traditionally modeled in terms of the cost and revenue functions.

The classic cost function gives the minimum cost of producing the target output *y*, given input prices *w*. The properties of cost function are the following

Theorem 3.3: *The cost function C is non-negative, non-decreasing, homogenous of degree one, concave, and continuous in prices w*.

Again, proofs of this theorem can be found in various sources, dating back to Shephard (1953) at least. But like in case of profit function, unnecessary convexity postulates are often imposed (just in case) to guarantee concavity of the cost function. It is worth stressing that these properties hold irrespective of convexity or non-convexity of the technology. As correctly noted by e.g. Färe and Primont (1995, pp. 44) and Russell (1998, pp. 46), it suffices that the images of the input correspondence *L* are closed.

For extracting technology information from the cost function, the input correspondence *L* proves the most natural dual partner. Utilizing direct analogy with Theorem 3.2, it is straightforward to prove the following.

Theorem 3.4: *The cost function C can recover the convex monotonic hull of the image of input correspondence L, that is,*

$$\left\{ x \in \mathbb{R}^r_- \middle| -w \cdot x \geq C(w,u) \ \forall w \geq \vec{0} \right\} = cm(L(u)) .$$

Proof. Follows directly from Proof 3.2 by fixing an arbitrary $u \in \mathbb{R}^s_+$.

Analogously, the revenue function *R* (Shephard, 1970) gives the maximum turnover achievable with a given vector of inputs *x* at the given output prices *p*. Assuming only that the images of output correspondence *P* are closed sets, the revenue function has the following properties:

Theorem 3.5: *The revenue function R is non-negative, non-decreasing, homogenous of degree one, convex, and continuous in prices p*.

For completeness and comparability, we characterize the following duality relationship, phrasing in terms of the output correspondence *P*:

Theorem 3.6: *The revenue function R can recover the convex monotonic hull of the image of output correspondence P, that is*

$$\left\{u \in \mathbb{R}_{+}^{s} \middle| v \cdot u \leq R(x,v)\ \forall v \geq \vec{0}\right\} = cm(P(x)).$$

Proof. Analogous to Proof 3.2.

In summary, we have noted that convexity is a completely redundant regularity property for constructing "well-behaved" economic models, which are convex (or concave) in dual variables (prices). Put differently, convexity has an asymmetric character in Duality Theory: We may need convexity for recovering the (exact) production model from an economic model, but not for recovering an economic model from a production model. In addition, there are differences in the technology information of different economic models. The cost and revenue functions have a much richer technology information than the profit function. For example, both these models can exhibit economies of scale. The cost function only fails to detect non-convexities and congestion in the input space. In turn, the revenue function is ' blind' to these properties in the output space. Thus, the technological information content of the cost and the revenue function also partly depends on the dimensionalities *r* and *s*: For single-output multiple-input technologies, for instance, the revenue function only fails to identify violations of free output disposability, and hence gives a better representation of the technology than the cost function. Moreover, the two models are complementary in the sense that the knowledge of *both* the cost function *and* the revenue function enables one to identify (at least some) violations of convexity both in the input and the output space, as well as in the full input-output space. However, the convex monotonic envelopments of *L* and *P* do not generally suffice for recovering the exact production possibility set *T*.

4. Indirect Models

Thus far we have only considered the most traditional production models which were defined solely in terms of input-output quantities, and the most basic economic models which included the possibility of quantity constraints. Besides quantity restrictions, the firm' s production choice may be constrained by monetary budget constraints for inputs, or minimum turnover limits for outputs. On the input side, this means that the scarcity does not concern the quantity of some specific input, but rather the financing of the entire input usage (or some part of it). For the outputs, this means that there is no specific quantity target. Instead, the firm is free to allocate its production to meet a certain financial turnover limit. [More general netput-based constraints will be considered in the next section.] This types of financial restrictions are usually modeled using *indirect* production and economic models, which are due to the work of Shephard (1974) and Shephard and Färe (1980).

This section extends the previous analyses to the indirect models. Although various duality relationships have been established *between* indirect models, and between indirect models and basic economic models, the genuine technology information of the indirect models has not been thoroughly investigated thus far.

The budget constraints of inputs are assumed to be of form $-w \cdot x \leq \beta$, where scalar $\beta \in \mathbb{R}_{+}$ denotes the budget, i.e., the maximum input expenditure allowed. Similarly, the turnover limits take the form $v \cdot u \geq \tau$, where scalar $\tau \in \mathbb{R}_{+}$ is the minimum acceptable revenue. Without loss of generality, these financial constraints can be expressed in terms of budget-normalized input prices as $-(w/\beta) \cdot x \leq 1$ or the turnover-normalized output prices as

$(v/\tau)\cdot u \geq 1$. Table 4 defines the indirect models to be considered.

Table 4: Indirect Models

Model	Formal definition
Revenue indirect input correspondence	$IL:\mathbb{R}_+^s \to 2^{\mathbb{R}_-^r}$, $IL(v/\tau) \equiv \left\{x \in \mathbb{R}_-^r \middle| (x,u)\in T; (v/\tau)\cdot u \geq 1\right\}$
Cost indirect output correspondence	$IP:\mathbb{R}_+^r \to 2^{\mathbb{R}_+^s}$, $IP(w/\beta) \equiv \left\{u \in \mathbb{R}_+^s \middle| (x,u)\in T; -(w/\beta)\cdot x \leq 1\right\}$
Revenue indirect input distance function	$ID_i : \mathbb{R}_-^r \times \mathbb{R}_+^s \to \overline{\mathbb{R}}_+$, $ID_i\,(x, v/\tau) \equiv \sup_{\theta,u}\left\{\theta \in \mathbb{R}_+ \middle| (x/\theta, u)\in T; (v/\tau)\cdot u \geq 1\right\}$
Cost indirect output distance function	$ID_o : \mathbb{R}_+^{r+s} \to \overline{\mathbb{R}}_+$, $ID_o(w/\beta, u) \equiv \inf_{\theta,x}\left\{\theta \in \mathbb{R}_+ \middle| (x, u/\theta)\in T; -(w/\beta)\cdot x \leq 1\right\}$
Revenue indirect cost function	$IC:\mathbb{R}_+^{r+s} \to \overline{\mathbb{R}}_+$, $IC(w, v/\tau) \equiv \sup_{x,u}\left\{-w\cdot x \middle| (v/\tau)\cdot u \geq 1; (x,u)\in T\right\}$
Cost indirect revenue function	$IR:\mathbb{R}_+^{r+s} \to \mathbb{R}_+$, $IR(w/\beta, v) \equiv \max_{x,u}\left\{v\cdot u \middle| -(w/\beta)\cdot x \leq 1; (x,u)\in T\right\}$

We next investigate whether the indirect models prove as good representations of the production possibilities as their direct counterparts?

First, the *revenue indirect input correspondence IL* (introduced in Färe and Primont, 1995) is the indirect version of the point-to-set mapping *L*, which indicates the set of input vectors that meet the indicated revenue target. Interestingly, the technology information content of *IL* corresponds to that of the revenue function, as demonstrated by the following theorem:

Theorem 4.1: *The technology information of the revenue indirect input correspondence IL is equivalent to the convex monotonic hull of the image of P. Specifically, the following equivalence holds for all* $(x,u)\in \mathbb{R}^{r+s}$

$$x \in \bigcap_{\{v/\tau | (v/\tau)\cdot u = 1\}} IL(v/\tau) \Leftrightarrow u \in cm(P(x)).$$

Proof. See Proof 4.1 in Appendix.

Analogously, the *cost indirect output correspondence IP* (Färe and Primont, 1995) is the indirect versions of the output correspondence *P*, which gives the set of output vectors expendable with the given budget.

Theorem 4.2: *The technology information of the cost indirect output correspondence IP is equivalent to the convex monotonic hull of the image of L. Specifically, the following equivalence holds for all* $(x,u)\in \mathbb{R}^{r+s}$

$$u \in \bigcap_{\{w/\beta | -(w/\beta)\cdot x = 1\}} IP(w/\beta) \Leftrightarrow x \in cm(L(u)).$$

Proof. Directly analogous to Proof 4.1.

Clearly these indirect point-to-set mappings offer a less detailed picture of the production possibilities than their direct counterparts. Specifically, they fail to identify violations of convexity in the netput dimensions subject to the financial constraints. These results are in some contrast with the conclusions of Färe and Primont (1995, p. 82, Figure 4.1), who suggest that the indirect output set $IP(w/\beta)$ "*is the outer envelope of the direct output sets* $P(x)$ *for which the cost of inputs does not exceed* [β, i.e., $-w \cdot x \leq \beta$]". This appears somewhat misleading, since $IP(w/\beta)$ actually spans an outer envelope the input sets, not the output sets. More precisely stated, $IP(w/\beta)$ is the union of such output sets $P(x)$ where x meets the budget constraint. Note that this union need not be convex, even if the images $P(x)$ themselves were convex.

The *revenue indirect input distance function* ID_i (Shephard, 1974) is the indirect version of the input distance functions D_i. It gives the maximum contraction of the given input vector x such that the indicated revenue target can be met. In other words, ID_i could be interpreted as the (direct) input distance measured to the indirect input set. Its technology information content is characterized by the following theorem:

Theorem 4.3: *The technology information of the revenue indirect input distance function* ID_i *is equivalent to the envelopment set characterized by weak disposability in the input space and convexity and free disposability in the output space. Specifically, the following equivalence holds for all* $(x,u) \in \mathbb{R}^{r+s}$

$$\min_{v/\tau} \left\{ ID_i(x, v/\tau) \middle| (v/\tau) \cdot u = 1 \right\} \geq 1 \Leftrightarrow x \in coni^{[1,\infty)}(L(u)) \wedge u \in cm(P(x)) .$$

Proof. See Proof 4.3 in Appendix.

The *cost indirect output distance function* ID_o (also by Shephard, 1974) is the indirect version of the output distance functions D_o, and hence a ' mirror image' of the previous model It indicates the maximum augmentation of the given output vector y affordable with the given budget. Analogous to Theorem 5.3, we propose the following:

Theorem 4.4: *The technology information of the cost indirect output distance function* ID_o *is equivalent to the envelopment set characterized by convexity and free disposability in the input space and weak disposability in the output space. Specifically, the following equivalence holds for all* $(x,u) \in \mathbb{R}^{r+s}$

$$\max_{w/\beta} \left\{ ID_o(w/\beta, u) \middle| -(w/\beta) \cdot x = 1 \right\} \leq 1$$

$$\Leftrightarrow x \in cm(L(u)) \wedge u \in coni^{[0,1]}(P(x))$$

Proof. Directly analogous to Proof 4.3.

Theorems 4.3 and 4.4 conform with the earlier results, and warrant no further comment.

Finally, also the traditional cost and revenue functions have their indirect counterparts: The revenue indirect cost function IC gives the minimum cost of producing an output bundle that meets the given turnover limit τ. It is straightforward to verify that the properties of IC are identical to the direct cost function (Theorem 3.3 above). However, its technological information content is more limited.

Theorem 4.5: *The technology information of the revenue indirect cost function IC is equivalent to the envelopment set characterized by convexity and free disposability both in the input and the output spaces, respectively. Specifically, the following equivalence holds for all* $(x,u) \in \mathbb{R}^{r+s}$

$$\max_{w,v/\tau}\left\{IC(w,v/\tau)\middle|-w\cdot x=1;(v/\tau)\cdot u=1\right\}\le 1$$
$$\Leftrightarrow x\in cm(L(u))\wedge u\in cm(P(x)).$$

Proof. See Proof 4.5 in Appendix.

Before commenting this result, we consider the parallel case of the *cost indirect revenue function IR*, which gives the maximum revenue achievable with the given budget β. It' s key properties coincide with those of the standard revenue function (Theorem 3.5). Interestingly, the indirect cost and revenue functions induce the identical envelopment sets, characterized by concave non-increasing input and output isoquants.

Theorem 4.6: *The knowledge of the revenue indirect cost function IC suffices for recovering the envelopment set characterized by convexity and free disposability both in the input and the output spaces, respectively. Specifically, the following equivalence holds for all* $(x,u) \in \mathbb{R}^{r+s}$

$$\max_{w/\beta,v}\left\{IR(w/\beta,v)\middle|-(w/\beta)\cdot x'=1;v\cdot u=1\right\}\le 1$$
$$\Leftrightarrow x\in cm(L(u))\wedge u\in cm(P(x)).$$

Proof. Directly analogous to Proof 4.5.

Inspired by the work of Petersen (1990), these envelopments with 'convexified' images of both *L* and *P* have attracted a lot of research interest in the recent literature of non-parametric Data Envelopment Analysis (DEA) method (see e.g. Bogetoft, 1996; and Bogetoft, Tama, and Tind, 2000).[16] However, the economic rationale of these "relaxed" convexity assumptions has been unclear (see e.g. Kuosmanen, 2001, for critical discussion). In this respect, Theorems 4.5 and 4.6 are very interesting results, because they establish a sound duality relationship between the Petersen technology and the indirect cost and revenue functions. For DEA, this reveals an intimate connection between Petersen' s non-convex approach and the indirect, cost/revenue constrained approach to economic efficiency measurement (e.g. Färe, Grosskopf, and Lovell, 1994), a connection which these two separately developed streams of literature could both benefit of. We believe this interesting link has gone unnoticed because the duality relationships between the indirect economic models on one hand and direct production models on the other have not been subject to a systematic study. For example, the remarkably comprehensive treatment of Färe and Primont (1995) only presents 13 duality theorems out of the possible 36 relationships available between 9 alternative models.

5. Constrained profit function

McFadden (1978, Part II) introduced the *restricted profit function*, a generalization of the

[16] The original article by Petersen also imposed a variety of alternative assumptions on returns to scale. For brevity, these returns to scale assumptions are omitted here.

basic economic models discussed in Section 4.[17] It extends the scope of the traditional cost and revenue functions in that the subset of fixed netput factors need not coincide with the subset of inputs or the subset of outputs. That is, any combinations of fixed and variable netputs may be modeled in terms of the restricted profit function. For some reason, both the theoretical and the applied streams of literature have ignored McFadden's contribution, although a large number of studies apply the general idea of (quasi-)fixed or non-discretionary netput variables.

I next propose to further generalize McFadden's notion by introducing a novel concept of *constrained profit function*. Let **A** denote a matrix with dimensionality $(r+s)\times k$, and use c for a k dimensional row vector, where k is some sufficiently large natural number.

Definition: *The constrained profit function is defined as the mapping*

$$\pi:\mathbb{R}_{+}^{m}\times\mathbb{R}^{k\times m+k}\rightarrow\overline{\mathbb{R}},\ \pi(p;\mathbf{A},c)\equiv \sup_{y}\left\{p\cdot y \,\middle|\, y\in T;\, y\mathbf{A}\leq c\right\}.$$

For any given **A** and c, the properties of the constrained profit function π coincide with those of the standard profit function Π (see Theorem 3.1).

The motivation of π is two-fold: The first motive is the operational convenience. Elements of matrix **A** and vector c can be viewed as *firm-specific* parameters reflecting internal *quantity* and/or *budget constraints*. The very simple inequality constraint $y\mathbf{A}\leq c$ proves a convenient structure for modeling a wide variety of constraint configurations - quantity restrictions[18] as well as monetary budget constraints and turnover targets.[19] The key advantage to earlier models (such as McFadden's notion) is that π does not require a priori specification of the fixed and the variable netputs, or the budget-constrained netputs. For example, the subsets of fixed netputs may differ from one firm to another.

The following examples illustrate the modeling possibilities allowed by this constraint. Denote row j of **A** by A_j.

I) If an arbitrary column of A (say column j) reads $A_j = (1\ 0\ \ldots\ 0)^T$ and $c_j = a$, the netput 1 is restricted to the half-closed interval $y_1\in(-\infty,a]$. In particular, if we set $c_j = 0$, then netput 1 must be an input ($y_1 \leq 0$).

II) Similarly, if $A_j = (-1\ 0\ \ldots\ 0)^T$ and $c_j = b$, the netput 1 falls to the half-closed interval $y_1\in[b,\infty)$. In particular, if we set $c_j = 0$, then netput 1 is an output ($y_1 \geq 0$).

III) Analogously, specifying $A_j = (1\ 0\ \ldots\ 0)^T$, $c_j = b$, and $A_{j+1} = (-1\ 0\ \ldots\ 0)^T$, $c_{j+1} = a$, $a<b$ we limit the values of netput 1 to the closed interval ($y_1\in[a,b]$). In particular, setting $c_j = c_{j+1} = a$ makes netput 1 a non-discretionary fixed factor $y_1 = a$.

IV) Suppose netputs 1 and 2 are inputs (see point I above), and their prices are p_1 and p_2 respectively. Specifying $A_j = (-p_1\ \ -p_2\ \ 0\ \ldots\ 0)^T$, $c_j = d$ we impose the budget constraint $p_1 y_1 + p_2 y_2 \leq d$.

The second motive is the generality. The constrained profit function binds together all alternative economic models considered thus far, including the indirect cost and revenue

[17] McFadden's notion has some evident similarities with the national profit function (also known as variable profit function) introduced by Samuelson (1953-1954).

[18] McFadden's restricted profit function could be modeled in a similar fashion by restricting matrix **A** to the class of diagonal matrices, and substituting the inequality constraint by an equality constraint.

[19] Inequality constraints of this type have proved a useful tool in the so-called 'weight-restricted' or 'assurance-region' nonparametric approaches (e.g. Allen *et al.*, 1997, for survey). The key difference to these approaches is that we here constrain the primal quantity variables, while the other approaches impose constraints on the dual variables: the prices or the marginal substitution rates.

functions. For completeness, Table 5 lists all these connections formally. These relations can be easily verified by inserting the indicated arguments to the constrained profit function. However, it is easy to show by simple counter-examples that none of the alternative economic models contains enough information for recovering the constrained profit function, except for special technologies.

Table 5: Special cases of the constrained profit function

Model	Relationship
Π	$\Pi(p) = \pi(p; \mathbf{0}, \vec{0})$
C	$C(w,u) = -\pi\left((w,\vec{0}); \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{pmatrix}, (u \quad -u) \right),\ k = 2s$
R	$R(x,v) = \pi\left((\vec{0},v); \begin{pmatrix} \mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, (x \quad -x) \right),\ k = 2r$
IC	$IC(w, v/\tau) = -\pi((w,\vec{0}); (\vec{0} \quad -v/\tau)^T, 1),\ k = 1$
IR	$IR(w/\beta, v) = \pi((\vec{0},v); (w/\beta \quad \vec{0})^T, 1),\ k = 1$

Of course, from the applied perspective the generality of this notion can also count as a handicap. Enormous amount of data would be needed for any serious empirical estimation of the maximum profit obtainable for alternative constraint configurations. Even though the data problems could be overcome by imposing additional (parametric) structure, the primary value of this notion does probably not come from empirical application. Still, we believe this general notion contributes to our understanding of the structure and the relationships of alternative economic models. It conveniently illustrates how the apparently different firm objectives such as cost minimization or revenue maximization can be understood as special cases of profit maximization under constraints. Like the directional distance function of Chambers et al., or McFadden's restricted profit function, the constrained profit function offers a 'super-model', a conceptual umbrella that covers a large number of standard models as its special cases, and thus serves as a ready platform for straightforward extensions and modifications. In this sense, the constrained profit function could provide useful guidance in model specification.

If nothing else, we find the general constrained profit function a convenient instrument for summarizing the role of constraints in determination of the duality. I think the following theorem aptly confirms the general conclusion from the preceding sections: The simpler the economic model, the stronger technology assumptions are required for establishing the duality.

Theorem 5.1: *The constrained profit function π contains the complete technology information of the production possibility set T. Specifically, the following equivalence holds for all $y \in \mathbb{R}^m$*

$$\exists \mathbf{A}, c : p \cdot y = \pi(p; \mathbf{A}, c)\ \forall p \Leftrightarrow y \in T$$

Proof. See Proof 5.1 in Appendix.

By Theorem 5.1, it is possible -at least in principle- to perfectly recover the production technology from the data of prices, profit, and constraint parameters (**A** and *c*), without any data on netput quantities. In particular, if we know the constrained profit function, we can test if a particular netput vector *y* is feasible by setting $\mathbf{A} = (\mathbf{I} \quad -\mathbf{I}), c = (y \quad -y)$. If $\pi(p, \mathbf{A}, c) = -\infty\ \forall p \in \mathbb{R}^m_+$, then the netput *y* is infeasible. Otherwise, [i.e., if a real valued profit is obtained for any prices at all] y must be feasible. In principle, we could recover the production possibility set T by means of testing feasibility of every $y \in \mathbb{R}^m$, but that would probably take an infinite time. On the other hand, that would be an entirely unnecessary exercise since π was shown to be a completely equivalent representation of technology.

In contrast to all other known duality results, Theorem 5.1 does not depend on any economically restrictive form of convexity or other topological conditions. This is essentially attributed to the fact that our generalized economic model allows for 'maneuvering' with the quantity/budget constraints [like the direct and indirect cost/revenue functions also do]. The proof of Theorem 5.1 just drives this possibility to its ultimate limit: By imposing certain constraints the feasible set becomes a singleton, which is a convex set. This result should be viewed against (and contrasted to) the body of established duality theory. By introducing a more general (or more 'flexible') economic model, in other words adding more information (which may be purely monetary price/profit data), we are able to reveal non-convexities of the technology, or alternatively stated, avoid the convexity and disposability axioms of the conventional Duality Theory.[20]

6. Discussion and Conclusions

The preceding sections have explored the role of convexity in representing technology by alternative production models, and recovering technology information from economic and indirect models defined in terms of monetary price and profit (cost/revenue) variables. It is now time to reflect back to the two objectives specified in the Introduction.

My first motivation was understanding the function of convexity in Duality Theory at a more profound level. To this end, the previous sections have paid special attention to the information content of alternative economic, production, and indirect models as representation of technology. This more "anatomical" perspective on Duality Theory aptly reveals the asymmetric character of convexity: economic models such as cost and revenue functions can always be derived from the technology, and they are well-behaved (concave/convex) irrespective of the topological shape of the technology. However, we may need some convexity properties for recovering exact representation of technology from these economic models.

There are considerable differences in the technology information of alternative models, which explains why the traditional duality theory requires a number of simplifying assumptions to establish equivalence between all models, i.e., to form of the "duality diamond". In general, the simpler the model, the stronger the (simplifying) assumptions required for establishing the duality. The static economic models of price-taking firm are conventionally enriched by quantity restrictions and/or financial constraints. Quite intuitively, these richer economic models also include more detailed information of the production possibilities. Interestingly, introducing the standard quantity/financial constraints to the usual setting of price-taking firm under certainty already suffices for identifying (at least some) non-convexities. The greater the 'variability' in constraints, the richer in detail the induced envelopment technology. By introducing the general notion of constrained profit function, we demonstrated that the traditional perspective of price-taking firm, enriched by additional

[20] This result is not particularly dependent on the special (non-standard) structure of the constrained profit function introduced above, but naturally extends to other economic models considered in the literature. In fact, the constrained profit function is actually unnecessarily 'general' for producing this effect: A similar duality theorem can be established between T and the McFadden (1978) restricted profit function, which is a genuine special case of our constrained profit function.

constraints, is sufficiently general to recover the complete and exact representation of technology.

For most applications of Duality Theory, equivalence of all models is not really necessary, or even desirable. To overcome data problems, for example, if the data does not suffice for estimating the desired target model, one would be mainly interested of what kind of information can be inferred from another model, which can be estimated. But this clearly does not necessitate equivalence of the two models involved, let alone all models. In this respect, the more anatomical perspective of this paper might provide useful guidance for parsimony with assumptions.

For completeness, Table 6 summarizes the specific conditions for recovering any of the models discussed in the preceding sections, given that another model is known. These conditions follow as direct corollaries of the well-known duality theorems, and those proved in this paper, so we here merely tabulate these conditions. We refer to the known model by 'base model' and to the model to be recovered by 'target model'. Each row of Table 6 corresponds to a specific base model, and each column to a specific target model. Having chosen the two models, we can read the necessary and sufficient conditions (abbreviated) from the cell corresponding to the row of the base model and the column of the target model. The empty cell indicates that the target model can be recovered from the base model without any additional assumptions. The distinction of inputs and outputs is assumed throughout. Consequently, the "global" convexity property $T = conv(T)$ is decomposed to three components: input convexity (ci), output convexity (co), and diseconomies of scale (des). Note that the conditions not only depend on the information content of the base model, but also that of the target model. For example, profit function can be recovered from any other model without any additional assumptions, because any limitations of the base model are cancelled out by the even stronger limitations of the profit function.

My second objective was to strengthen the foundation of the fundamental Duality Theory by generalizing it further towards non-convex technologies. Although the mathematical importance of convexity in Duality Theory is undeniable, we were able to eliminate (or at least alleviate) the economically restrictive convexity postulates by drawing a sharp distinction between the technology sets and their outer bound approximations. For convex technologies, there is no approximation error involved, because the outer approximation and the true technology set coincide. Hence, the established results of Duality Theory follow conveniently as the special cases of the more general 'envelopment duality' framework. Most interestingly, we have demonstrated that Duality theory can also be applied to non-convex technologies. Occurrence of non-convexities (or congestion) simply means that the outer envelopment is an inaccurate approximation. Still, the outer envelopment is typically better than nothing!

In fact, in some applications the outer approximation is all we need to care about. Consider for example the measurement of cost efficiency when we have data on total costs, input prices, and output quantities for a sample of n firms. Using duality theory, we can estimate the cost function and we can analyze cost efficiency without any data on input quantities. What seems not too evident from traditional Duality Theory is that potential non-convexities do not matter! We do not need any to assume any form of convexity for applying Duality Theory for measuring cost efficiency by price/cost data. Of course, there also are applications where the outer approximation does not suffice. For example, the classic Farrell (1957) decomposition of the cost efficiency index into components of allocative efficiency and technical efficiency fails unless the image of the input correspondence L is a convex set for all output vectors y.

Table 6: The conditions for recovering the target model (on the 1st row) given the base model (on the 1st column).

		TARGET MODEL															
		T	L	P	D_i	D_o	DD	Π	C	R	IL	IP	ID_i	ID_o	IC	IR	π
BASE MODEL	T																
	L																
	P																
	D_i	wdi	wdi	wdi		wdi	wdi			wdi	wdi			wdi			wdi
	D_o	wdo	wdo	wdo	wdo		wdo		wdo			wdo	wdo				wdo
	DD	dd	dd	dd	dd	dd											dd
	Π	des ci co fdi fdo	des ci co fdi fdo	des ci co fdi fdo	des ci co fdi fdo	des ci co fdi fdo	des ci co fdi fdo		des co fdo	des ci fdi	des ci fdi	des co fdo	des ci fdi	des co fdo	des	des	des ci co fdi fdo
	C	ci fdi	ci fdi	ci fdi	ci fdi	ci fdi	ci fdi			ci fdi	ci fdi		ci fdi				ci fdi
	R	co fdo	co fdo	co fdo	co fdo	co fdo	co fdo		co fdo			co fdo		co fdo			co fdo
	IL	co fdo	co fdo	co fdo	co fdo	co fdo	co fdo		co fdo			co fdo		co fdo			co fdo
	IP	ci fdi	ci fdi	ci fdi	ci fdi	ci fdi	ci fdi			ci fdi	ci fdi		ci fdi				ci fdi
	ID_i	co fdo wdi	co fdo wdi	co fdo wdi	co fdo wdi	co fdo wdi	co fdo wdi		co fdo	wdi	wdi	co fdo		co fdo			co fdo wdi
	ID_o	ci fdi wdo	ci fdi wdo	ci fdi wdo	ci fdi wdo	ci fdi wdo	ci fdi wdo		wdo	ci fdi	ci fdi	wdo	ci fdi				ci fdi wdo
	IC	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo		co fdo	ci fdi	ci fdi	co fdo	ci fdi	co fdo			ci co fdi fdo
	IR	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo	ci co fdi fdo		co fdo	ci fdi	ci fdi	co fdo	ci fdi	co fdo			ci co fdi fdo
	π																

legend	**assumption**	**formal definition**
des	Diseconomies of scale	$x \in conv(L(u)) \wedge \; u \in conv(P(x)) \Rightarrow (x,u) \in conv(T)$
ci	Input convexity	$L(u) = conv(L(u)) \; \forall u \in \mathbb{R}^s_+$
co	Output convexity	$P(x) = conv(P(x)) \; \forall x \in \mathbb{R}^r_-$
fdi	Free disposability of inputs	$L(u) = L(u) - \mathbb{R}^s_+ \; \forall u \in \mathbb{R}^s_+$
fdo	Free disposability of outputs	$P(x) = P(x) - \mathbb{R}^r_+ \; \forall x \in \mathbb{R}^r_-$
wdi	Weak disposability of inputs	$L(u) = coni^{[1,\infty)}(L(u)) \; \forall u \in \mathbb{R}^s_+$
wdo	Weak disposability of outputs	$P(x) = coni^{[0,1]}(P(x)) \; \forall x \in \mathbb{R}^r_-$
dd	Directional disposability	$DD(y,g) \geq 0 \Rightarrow y \in T$

In my experience, convexity tends to be over-used in empirical applications, both in parametric/econometric as well as non-parametric/mathematical-programming approaches. However, the practical implications are somewhat different. The main difference seems not to be in the estimation technique per se, but the primal versus dual orientation of analysis. For the parametric approach which often operates on the dual side, the confusion with convexity tends to concentrate on the scope of analysis (i.e., results apply even in case of non-convex technology) and the interpretation of obtained results, not so much on the estimation results themselves. Typically, the parametric approach favors flexible function forms, which as such, do not restrict to any specific convexity or concavity properties. Therefore, Duality Theory is often -and quite correctly- used as a guideline for analyzing, testing, and enforcing the curvature properties of economic models.[21] The fact worth re-emphasizing in this context is that concavity of cost functions is an economic property independent of the underlying technology. Hence, if the estimated flexible-function-form cost function fails to be concave, non-convexities are not to blame.

On the other hand, parametric estimations often tend to restrict on the most standard economic models with well-known duality properties, cost functions in particular, even though some other model could be a more appropriate description of the economic objectives. I hope the comprehensive anatomical perspective and the set of "new" duality results of this paper might encourage more extensive application of the indirect/constrained models.

The non-parametric literature typically tends to operate with the primal-side quantity data, where price information sometimes has a complementary role. I am not aware of too many primarily price-based non-parametric analyses. Introduction section already acknowledged influence from the more application oriented Afriat-school and the Charnes-Cooper school of thought, which employ similar topological envelopments in empirical production analysis. The implications slightly differ between these two paradigms.

The Afriat-school is mainly concerned with testing optimization hypotheses, but also technical properties like convexity and free disposability (see e.g. Hanoch and Rothschild, 1972). Provided that the data conforms with optimization hypotheses, the Afriat-approach can proceed with predicting firm' s response to changed economic conditions, like price changes. This line of reasoning has been pursued most notably by Varian (1984). Traditionally, this approach assumes data of input-output quantities (at least). Our results imply that one could equally well adopt the dual perspective, and proceed with the monetary data. For testing convexity hypotheses, it does not really matter if we have quantity or monetary data. The possibility of identifying non-convexities essentially depends on the existence of binding quantity or financial constraints, like the previous analysis demonstrated. For example, consider forecasting the response of the competitive firm to regulation [i.e. introduction of quantity/budget constraints], in the spirit of Fulginiti and Perrin (1993). More specifically, suppose the constrained profit function (or one of its special cases) would be the relevant model to use, but we only have data available from the present unconstrained situation that only allows us to estimate the unconstrained profit function. In this case, the production possibility set would have to be truly convex in order to derive the constrained profit function from the unconstrained profit function, because the unconstrained profit function can only recover the convex monotonic hull of the production possibility set. Furthermore, analyzing quantity data instead of monetary data does not necessarily improve the situation, because there is an economic selection effect that should be taken into account. Under free competition, it is clear that no firm will find it optimal to operate in a non-convex region of the frontier. Therefore, *any data* collected prior to regulation is likely to conceal the non-convexities. Of

[21] In this respect, it is interesting to note that the parametric literature emphasizes the easier availability and reliability of price data (e.g. Young, Mittelhammer, and Rostamizadeh, 1985), while the non-parametric literature uses exactly the same arguments in favor of quantity data (e.g. Charnes and Cooper, 1985).

course, during regulation [i.e., presence of constraints] non-convexities may be of decisive importance.

The Charnes-Cooper school focuses on efficiency aspects in a sample of comparable organizational units. For efficiency analysis, the implications of Duality Theory mainly concern the specification of the maintained production assumptions. Traditionally, convexity plays an especially critical role (see e.g. Deprins, Simar, and Tulkens; and Tulkens, 1993). Since the prior knowledge of technological facts governing the production possibilities is typically very limited, the plausibility of competitive markets and price-taking behavior, at least by a reasonable empirical approximation, appears to be a very frequently used argument for imposing some form of convexity in this literature. This line of reasoning is sometimes presented explicitly (e.g., Thrall, 1999), but more often than not, the argument implicitly guides the model specification towards the convex alternatives. Although a wide variety of alternative non-convex or ' partially' convex model specifications are available (e.g. Petersen, 1990; Bogetoft, 1996, Bogetoft et al., 2000), the fully convex production possibility set remains the most frequently applied model specification in reported applications. However, our detailed review of alternative models reveals that most of the standard economic models do not offer any dual justification for such a strong maintained assumption. For measuring *profit efficiency* (e.g. Chambers, et al., 1998) when price data is missing, Duality Theory does justify omitting non-convexities of the technology.[22] For any other notion of efficiency, potential non-convexities call upon attention. For pure technical efficiency, for example, Duality Theory only forwards some minimal conditions of weak disposability. For these situations, the justification of maintained convexity assumptions must come from some other source. When the appropriate behavioral assumptions are chosen, Table 6 can provide valuable guidance in model specification, identifying the critical as well as truly harmless assumptions.

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[22] See Cherchye et al. (2000) for further discussion of this point.

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Appendix: Proofs

Proof 2.2: By straightforward calculus,

$x \in coni^{[1,\infty)}(L(u))$ (by definition of $coni^{[.)}(.)$)

$\Leftrightarrow x \in \lambda L(u)\,,\ \lambda \geq 1$

$\Leftrightarrow \exists \lambda \geq 1:\ x/\lambda \in L(u)$

$\Leftrightarrow \max\left\{\theta \middle|\ x/\theta \in L(u)\right\} \geq 1$ (definition of D_i)

$\Leftrightarrow D_i(x,u) \geq 1$

Proof 2.4:

$\min_{g \in \mathbb{R}^m_{++}} DD(y;g) \geq 0$

$\Leftrightarrow DD(y;g) \geq 0 \quad \forall g \in \mathbb{R}^m_{++}$

$\Leftrightarrow y \in \bigcap_{g \in \mathbb{R}^m_{++}} \left\{ y \in \mathbb{R}^m \middle| DD(y;g) \geq 0 \right\}$ (by definition of DD)

$\Leftrightarrow y \in \bigcap_{g \in \mathbb{R}^m_{++}} \left\{ y \in \mathbb{R}^m \middle| y = y' - \theta g;\, y' \in T; \theta \geq 0 \right\}.$

Proof 3.2:

$cm(T) \ = \bigcap_{\left\{p \in \mathbb{R}^m_+ \middle| p \cdot y' \leq 1\ \forall y' \in cm(T)\right\}} \left\{ y \in \mathbb{R}^m \middle| p \cdot y \leq 1 \right\}$ (applying Minkowski' s theorem)

$= \left\{ y \in \mathbb{R}^m \middle| p \cdot y \leq \sup_{y' \in cm(T)} (p \cdot y')\ \forall p \in \mathbb{R}^m_+ \right\}$ $\left(\max_{y' \in cm(T)} (p \cdot y') = \max_{y' \in T} (p \cdot y')\ \forall p \in \mathbb{R}^m_+ \right)$

$= \left\{ y \in \mathbb{R}^m \middle| p \cdot y \leq \sup_{y' \in T} (p \cdot y')\ \forall p \in \mathbb{R}^m_+ \right\}$ (definition of Π)

$= \left\{ y \in \mathbb{R}^m \middle| p \cdot y \leq \Pi^T(p)\ \forall p \in \mathbb{R}^m_+ \right\}$

Proof 4.1:

$x \in \bigcap_{\left\{v/\tau \middle| (v/\tau) \cdot u = 1\right\}} IL(v/\tau)$ (definition of IL).

$\Leftrightarrow \exists u' \in (P(x)) : (v/\tau) \cdot u' \geq 1$ for all $v/\tau \in \mathbb{R}^s_+ : (v/\tau) \cdot u = 1$ (multiply by τ)

$\Leftrightarrow \exists u' \in (P(x)) : v \cdot u' \geq 1$ for all $v \in \mathbb{R}^s_+ : v \cdot u = 1$

$\Leftrightarrow v \cdot u \leq \max_{u' \in P(x)} (v \cdot u')\ \forall v \in \mathbb{R}^s_+$ (applying Minkowski' s theorem)

$\Leftrightarrow u \in cm(P(x))\,.$

Proof 4.3:

$x \in coni^{[1,\infty)}(L(u)) \wedge u \in cm(P(x))$ (the latter condition: Minkowski' s theorem)

$\Leftrightarrow x \in \lambda L(u), \lambda \geq 1\ \wedge\ v \cdot u \leq \max_{u' \in P(x)} (v \cdot u')\ \forall v \in \mathbb{R}^s_+$ (normalize prices)

$\Leftrightarrow x \in \lambda L(u), \lambda \geq 1\ \wedge\ \exists u' \in P(x) : (v/\tau) \cdot u' \geq 1\ \forall (v/\tau) \in \mathbb{R}^s_+ : (v/\tau) \cdot u = 1$ (definition ID_i)

$\Leftrightarrow ID_i(x, v/\tau) \geq 1\ \forall v/\tau : (v/\tau) \cdot u = 1$

$\Leftrightarrow \min_{v/\tau} \left\{ ID_i(x, v/\tau) \middle| (v/\tau) \cdot u = 1 \right\} \geq 1$

Proof 4.5:

$x \in cm(L(u)) \wedge u \in cm(P(x))$

$\Leftrightarrow w \cdot x \geq \min_{x' \in L(u)} (w \cdot x') \ \forall w \in \mathbb{R}^r_+ \wedge v \cdot u \leq \max_{u' \in P(x)} (v \cdot u') \ \forall v \in \mathbb{R}^s_+$

$\Leftrightarrow \max_{u' \in P(x)} ((v/\tau) \cdot u') \geq 1 \ \wedge \min_{x' \in L(u)} (w \cdot x') \leq 1 \ \ \forall w, v/\tau : w \cdot x = 1, (v/\tau) \cdot u = 1$.

$\Leftrightarrow \exists u' \in P(x) : (v/\tau) \cdot u' \geq (v/\tau) \cdot u \ \wedge \min_{x' \in L(u)} (w \cdot x') \leq 1 \ \ \forall w, v/\tau : w \cdot x = 1, (v/\tau) \cdot u = 1$ (definition, *IC*)

$\Leftrightarrow IC(w, v/\tau) \leq 1 \ \forall w, v/\tau : w \cdot x = 1, (v/\tau) \cdot u = 1$

$\Leftrightarrow \max_{w, v/\tau} \left\{ IC(w, v/\tau) \middle| w \cdot x = 1; (v/\tau) \cdot u = 1 \right\} \leq 1$.

Proof 5.1: Organized in two parts:

Part $\Leftarrow$: Consider an arbitrary netput vector $y' \in T$. We can set $\mathbf{A} = \begin{pmatrix} \mathbf{I} & -\mathbf{I} \end{pmatrix}$, and $c = (y' \quad -y')$. Since for this $\mathbf{A}$ and c the vector y' is the only feasible point that satisfies the inequality $y\mathbf{A} \leq c$, we obviously have $\pi(p; \mathbf{A}, c) = p \cdot y' \ \forall p$, which confirms the first part.

Part $\Rightarrow$: $y'' \in \mathbb{R}^m$, suppose there exist $\mathbf{A}$ and c such that $\pi(p, \mathbf{A}, c) = p \cdot y'' \ \forall p \in \mathbb{R}^m_+$. The condition "for all *p*" is the key: For any pair of non-identical non-negative input-output vectors $y, y'; y \neq y'$, there exist *some* non-negative prices *p* such that $p \cdot y \neq p \cdot y'$. This implies the inequality constraint $y\mathbf{A} \leq c$ is only satisfied by a single point, i.e., $\left\{ y \in \mathbb{R}^m \middle| y\mathbf{A} = c \right\} = \left\{ y'' \right\}$. Now, suppose that $y'' \notin T$. This would imply $\pi(p; \mathbf{A}, c) = -\infty \ \forall p$. which leads to a contradiction: However, any non-negative y'' yields a finite profit at some non-negative prices. Hence, we must have $\pi(p, \mathbf{A}, c) \neq p \cdot y''$ for some non-negative prices *p*. Therefore, assumption $y'' \notin T$ leads to a contradition. By reductio ad absurdum, we must have $y'' \in T$.

Combining the two parts proves the equivalence.