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AN ASSESSMENT OF CONVEXITY AXIOMS IN DEA*

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ABSTRACT
Data Envelopment Analysis (DEA) is traditionally based on the axiom of convex production possibility sets. In many research situations this axiom is considered overly restrictive, and recent research has focused on finding suitable ways of relaxing it. Unfortunately, there currently is no consensus in the DEA field on why and how to account for non-convexities. This paper explores empirical evidence as well as theoretical and practical arguments in favor of and against convexity axioms in DEA. These considerations are of key importance for model specification in practical applications, as well as for directing future research on DEA.

KEYWORDS: Data Envelopment Analysis (DEA), Non-parametric production analysis, Convexity.

1. INTRODUCTION

Data Envelopment Analysis (DEA; Charnes *et al.*, 1978, Banker *et al.*, 1984, Charnes *et al.*, 1985) is a mathematical programming technique for efficiency evaluation. Building on some basic axioms from activity analysis (Dantzig, 1949, and Koopmans, 1951), DEA constructs a subset of the input-output space that envelops all observed Decision Making Units (DMUs). Following Farrell (1957), DEA measures efficiency of DMUs relative to the boundary of that set. In economic terminology, this envelopment set can be viewed as an empirical approximation of the (closed and non-empty) *production possibilities set*

(1) $$T = \{(x, y) \in \Re_+^{s+m} \,|\, x \text{ can produce } y\},$$

where $x \in \Re_+^m$ denotes an input vector and $y \in \Re_+^s$ denotes an output vector. At due time, we will use two alternative technology representations: the *input set*

(2) $$L(y) = \{x \,|\, (x, y) \in T\} \quad y \in \Re_+^s,$$

and the *output set*

(3) $$P(y) = \{y \,|\, (x, y) \in T\} \quad x \in \Re_+^m.$$

Different empirical production sets can be derived from different production axioms. The original DEA model by Charnes *et al.* (1978) is based on the following axioms[2]:

(A1) *monotonicity*, i.e. $T = T + \Re_+^m \times \Re_-^s$.

(A2) *convexity*, i.e. $T = c(T)$, where $c(\cdot)$ denotes the convex hull.

[2] Activity analysis frequently uses the more elementary axioms of *divisibility*, i.e. $(x, y) \in T \quad \lambda(x, y) \in T \quad \forall \lambda \in [0,1]$, and *additivity*, i.e. $(x, y),\ (x', y') \in T \quad (x, y) + (x', y') \in T$ to justify convexity (see e.g. Arrow and Hahn, 1971). Together, these two axioms are equivalent to the axioms of convexity and ray unboundedness, i.e. both sets of axioms imply the production possibilities set is a convex cone.

(A3) *ray unboundedness*, i.e. $T = \lambda T, \lambda > 0$.

These axioms have the following economic interpretations: monotonicity means that all inputs and outputs are freely (or strongly) disposable; convexity is equivalent to decreasing marginal rates of substitution (between inputs, between outputs and between inputs and outputs); finally, ray unboundedness means that *T* exhibits constant returns-to-scale.

DEA is often credited for using a minimal set of assumptions, and for letting the data speak for itself rather than enforcing it to the idiom of some arbitrarily specified parametric structure. Therefore, the motivation for imposing the original set of production axioms is of direct interest. Indeed, relaxation of these axioms has attracted considerable attention in the DEA literature.

Monotonicity (A.1) is sometimes debatable because it excludes *congestion*, which is frequently observed e.g. in agriculture and transportation, as pointed out by Färe and Svensson (1980) and Färe and Grosskopf (1983). Ways of dealing with congestion (by weakening or dropping the monotonicity axiom) have been proposed in the DEA literature (most notably by Färe and Grosskopf 1983; Färe, Grosskopf, and Lovell, 1983, 1985; and Brocket *et al.*, 1998). We will not discuss these in detail in this paper, but refer to Cherchye *et al.* (2001a) for a recent assessment of congestion analysis within DEA.

Ray unboundedness (A.3) is frequently considered overly restrictive, because many production activities exhibit increasing and/or decreasing returns to scale (see e.g. Farrell, 1957, for early accounts). Färe, Grosskopf and Logan (1983), Banker *et al.* (1984), Grosskopf (1986) and Seiford and Thrall (1990), among others, have discussed implementation of alternative returns-to-scale axioms in DEA.

Recent research has paid considerable attention to relaxing the convexity axiom (A.2). For example, Deprins *et al.* (1984) and Tulkens (1993) dropped convexity altogether. Petersen (1990) and Bogetoft *et al.* (2000) replaced convexity of *T* with the somewhat milder assumption of convexity of input and output sets, i.e. $L(\cdot) = c(L(\cdot))$ and $P(\cdot) = c(P(\cdot))$. Next, Bogetoft (1996), Chang (1999), and Post (2001a) have considered convexity of either input sets $L(\cdot)$ or output sets $P(\cdot)$, but not both. Finally, Post (2001b) and Kuosmanen (2001) replaced convexity by the modified properties of '*transconvexity*' and '*conditional convexity*' respectively.

Unfortunately, there currently is no consensus on why or how to allow for non-convexities in DEA. Some authors firmly hold that convexity is not restrictive at all. For example, Thrall (1999, p. 244) states: "*avoidance of convexity is not compatible with a real world economy where values (prices and costs) are considered important. Alternatively stated, real world economics brings convexity in through the back door even though it is denied entrance at the front.*" In addition, the literature forwards many arguments that are indistinct or even demonstrably incorrect. For example, Petersen (1990, pp. 307) denied the possibility of non-convex input or output sets by referring to "*the law of diminishing marginal rates of substitution*", which to our knowledge is not documented as a law in microeconomic production theory.

In this paper, we will systematically explore existing arguments and evidence in favor of and against convexity axioms in DEA. We will also present some new arguments that refute widespread misconceptions concerning the convexity axioms. These points have implications for model specification in practical applications. In addition, they provide guidelines for the further theoretical development of DEA.

The remainder of this paper is organized as follows. Section 2 discusses possible sources of violations of convexity from a theoretical point of view. Section 3 reviews empirical evidence for such violations as reported in three distinct traditions of empirical production analysis. Section 4 assesses theoretical arguments for convexity axioms within economic and technical efficiency analyses. Section 5 discusses practical arguments that originate from the need to estimate an empirical production set from a finite set of observations. Finally, Section 6 summarizes our findings and points at a number of interesting avenues for further research.

2. SOURCES OF NON-CONVEXITIES

Some authors have referred to microeconomic production theory to justify the convexity axioms imposed in DEA. However, microeconomic production theory only forwards very weak guidelines for modeling the physical production possibilities, like the fundamental notion of *scarcity*, which in this context implies that not all input-output combinations are feasible (and hence introduces the need to specify technological constraints in the first place). For example, to the best of our knowledge, the “*law of diminishing marginal rates of substitution*”, as referred to by Petersen (1990), Bogetoft (1996), and Bogetoft *et al.* (2000) as a justification of convex input and output sets, is not documented in microeconomic production theory. In fact, non-convexities play an important role in economic theory, as we will demonstrate in the following.

The convexity axiom (A2) assumes away (1) indivisible inputs and outputs, (2) economies of scale, and (3) economies of specialization (=diseconomies of scope). The economic importance of these phenomena was already stressed by Farrell in his famous 1959 article "The Convexity Assumption in the Theory of Competitive Markets", Section II (entitled 'The importance of non-convexities'):

> 'A glance at the world about us should be enough to convince us that most commodities are to some extent indivisible and that many have large indivisibilities. Similarly, whenever one refers to "economies of scale" or of "specialization", one is pointing to concavities [=departures from convexity (CKP)] in production functions. There is thus no need to argue the importance of either indivisibilities or concavities in production functions - the former are an obvious feature of the real world, and the latter have constituted a central topic in economics since the time of Adam Smith.'
>
> Farrell (1959, pp. 378 – 379)

Convexity requires that all inputs and outputs are divisible. As Farrell notes, however, most commodities are clearly indivisible, which immediately violates convexity. Still, for some input or output commodities, the indivisible units are so small that divisibility of commodities is a useful approximation (say sugar). In general, it seems that divisibility of commodities is a more reasonable approximation at the industry level or at the level of the entire economy (where commodities are usually measured in thousands or millions) than at the firm or plant level.

Divisibility of inputs and outputs does not yet imply convexity, as convexity requires the production activities themselves to be divisible, at least to some extent. Therefore, even if all inputs and outputs are perfectly divisible, *economies of scale* and *of specialization* can still violate convexity. While these notions are clearly related, they are not equivalent. Specifically, economies of scale pertain to increasing marginal productivity (implying non-convex production sets), while economies of specialization are associated with decreasing substitution rates of inputs (implying non-convex input sets) and increasing transformation rates of outputs (implying non-convex output sets). See e.g. Yang and Ng (1993) for further discussion.

For a long time, economists have referred to economies of scale as one important explanation for benefits from trade (see e.g. Dixit and Stiglitz, 1977; Ethier, 1979; Krugman, 1980; and Grossman and Helpman, 1989). The recent literature has stressed the role of economies of specialization in many areas of economics, including the theory of international trade (Yang, 1994); product development, evolution of the institution of firm, and economic growth (Yang and Ng, 1995; Borland and Yang, 1995); as well as emergence of industrialization and urban cities (Yang and Rice, 1994; Shi and Yang, 1995).

Sources of scale and scope economies can be found e.g. in economies of information processing (see e.g. Wilson, 1975, for an early treatment, and Mitchell, 2000, for further discussion and references); aspects like learning, communication, and coordination can favor specialization in a limited number of tasks and techniques. In addition, physical phenomena can directly entail violations of convexity, as the following example illustrates:

Consider construction of storage containers, say cylinder shaped oil containers. If the dimensions of the cylinder are chosen to minimize the surface area, the surface area A depends on the volume of the container V by the formula $A = \frac{3}{2\pi}V^{2/3}$. Typically, the material cost C (used as an input proxy) of the container is (roughly) directly proportional to the surface area, and hence is given by: [3]

(4) $$C = aV^{2/3},$$

where $a>0$ is a constant that combines both technical coefficients and price components. Hence, material cost is a non-convex function of volume, and the production possibilities set (i.e. the epigraph of the cost function) is non-convex. Interestingly, in chemical engineering, numerous studies following Chilton (1950) have found that the above theoretical two-thirds cost-capacity factor also applies as a reasonable approximation at the plant level cost-to-capacity analysis (see e.g. Ellsworth, 1998, for further discussion and references).

[3] This result is not limited to the geometry of cylinders in particular. Storage containers of other shapes (such as cubes, cones or balls) also have the cost-capacity factor of 2/3.

Economies of scale can also give rise to economies of specialization, and introduce non-convexity for the input sets and/or output sets. To illustrate this point, consider the material cost for storing two different petroleum qualities (say motor fuel and diesel oil). Of course, the two petroleum qualities cannot be mixed in the storage, so each of the two qualities requires a separate container. Assuming that material cost is directly proportional to the surface area of the two containers, it can be expressed as a function of motor fuel volume V_{MF} and diesel oil volume V_{DO}:

(5) $$C = aV_{MF}^{2/3} + bV_{DO}^{2/3} \quad a, b > 0 .$$

Clearly, this function is non-convex. In addition, the function is not quasi-convex, so that the output sets (i.e. the upper level sets of the material cost function) are not convex.[4] In this example, economies of scale in storage of each petroleum quality imply an incentive to specialize in only one quality (either motor fuel or diesel oil) in activities involving petroleum storing such as oil refining and petroleum transportation.

3. EMPIRICAL EVIDENCE

The prevalence of indivisible inputs and outputs needs no further comment. In the following we focus on economies of scale and of specialization as sources of non-convexity. These phenomena have been identified in empirical analyses of a wide variety of industries. The empirical evidence comes from at least three different traditions of empirical production analysis: 1) process analysis, 2) parametric estimation, and 3) non-parametric inference. Below we present a selection of examples from each of these traditions.

Process analysis

Process analysis derives production relations directly from theoretical and practical engineering knowledge. Many studies in this field have found evidence of violations of convexity. Especially economies of scale are well documented. Chenery (1949) pioneered the study of engineering production functions that allow for inference of scale and substitution properties. Investigating the pipeline transportation of natural gas, Chenery derived a (non-linear) cost function that exhibits economies of scale. As for other industries, Wibe (1984) surveyed 28 studies of engineering production functions mainly undertaken in manufacturing industries. The majority of the studies reporting scale elasticities provide evidence of increasing returns. In addition, economies of scale have been reported e.g. in chemical industries, manufacturing of process equipment, air pollution control equipment, and biopharmaceutical equipment, as well as in waste-to-energy facilities (see e.g. Ellsworth, 1998).

The process analytical approach can provide very convincing evidence, as it builds directly on engineering knowledge of the production process. Nevertheless, the evidence mostly comes from relatively simple activities due to major difficulties in

---

[4] Quasi-convexity requires the second-order bordered leading principal minor (i.e. the determinant of the bordered Hessian matrix) $2\left(\frac{\partial^2 C}{\partial V_{MF}\partial V_{DO}}\right)\left(\frac{\partial C}{\partial V_{MF}}\right)\left(\frac{\partial C}{\partial V_{DO}}\right)-\left(\frac{\partial^2 C}{\partial V_{MF}^2}\right)\left(\frac{\partial C}{\partial V_{DO}}\right)^2-\left(\frac{\partial^2 C}{\partial V_{DO}^2}\right)\left(\frac{\partial C}{\partial V_{MF}}\right)^2=\frac{8}{81}ab\left(bV_{MF}^{-1\frac{1}{3}}V_{DO}^{-\frac{2}{3}}+aV_{MF}^{-\frac{2}{3}}V_{DO}^{-1\frac{1}{3}}\right)$ to be negative for all $V_{MF}, V_{DO} \geq 0$.

transforming and aggregating the process specific engineering variables -such as temperature, pressure, and concentration levels- to inputs and outputs the firm transacts in its economic face. This may also partially explain why engineering production functions are so rarely reported.

Parametric estimation

The parametric approach calibrates the parameters of a regression function with a particular functional form, so as to provide the best fit to the observed production data (see e.g. Bauer, 1990, for a survey parametric production frontiers). Typically, the functional form is selected ad hoc without any reference to engineering knowledge, which casts some doubt on the results obtainable by this method. Still, the parametric method is applied very frequently.

Noteworthy findings of economies of scale and specialization have been reported within this tradition. For example, Hasenkamp (1976) studied the railroad industry data of the classic Klein (1947) study. Hasenkamp estimated Marshallian representative firm production functions using different functional forms (including standard Cobb-Doublas and Constant-Elasticity-of-Substitution (CES) formulations). For all functional forms considered, his estimation results indicated economies of scale, and for flexible enough functional forms (e.g. CES) they also reveal economies of specialization (i.e. violation of convexity in output space). Interestingly, Hasenkamp points out some important policy implications of this result:

> "In the past the railroad industry tried to drop passenger service from its outputs. This is precisely the behavior one would expect for a non-convex output function. … Non-convexity of the output function implies that public regulation should also apply to the mix of outputs, if both outputs – namely freight service and passenger service – are desired by the public."
>
> Hasenkamp (1976, pp. 261)

This is a relevant point to bear in mind in DEA applications where some DMUs are committed to a wide mix of output (or input) due to external constraints such as governmental regulation, while other DMUs can freely exploit economies of specialization.

Non-parametric estimation

The non-parametric approach (e.g. Farrell, 1957; Afriat, 1972; and Varian, 1984), which also includes DEA, does not assume a particular functional form but uses more general production axioms. Unfortunately, the convexity axiom has rarely been exposed to empirical tests within this tradition. There are at least three possible reasons. First, the existing specification tests are unsatisfactory for this purpose, because they can confuse non-convexities with the effect of small sample error (see Section 5 for further discussion). Second, convexity generally is not a property of direct interest to economists or operations researchers, although it does interfere with efficiency measures as well as phenomena like economies of scale and specialization, which are of direct interest. Third, we suspect that the 'established' status of convexity as a standard, theoretically sound regularity property may have discouraged empirical testing.

Still, a number of DEA studies have been reported that do assess convexity, e.g. Deprins *et al.*, (1984), Tulkens (1993), Kuosmanen (1999), and Dekker and Post (2001). Most of these studies examined the hypothesis of full convexity of the production set by comparing the convex Banker *et al.* (1984) model with the convexity-free Deprins *et al.* (1984) model (see Section 4 for further discussion). In all these studies, the convexity postulate had a major influence on the efficiency results. For example, in the retail banking application reported by Tulkens (1993), 74.6% of public bank branches were classified as efficient under a non-convex approximation, whereas only 5.2% of these branches remained efficient when the convexity assumption was added. For private bank branches, the same study identified 57.8% efficient branches under the non-convex approximation but only 5.5% efficient ones under the convex approximation. In another study on bank branch data, Dekker and Post (2001) also found substantial deviations between the results for the two models. Both studies involved a sample size that is much larger than in typical applications. Therefore, it seems unlikely that these results could be explained solely by small sample error.

To conclude, in light of the empirical evidence presented above, there is no good reason for considering convexity of production sets or convexity of input/output sets as generally realistic axioms. The evidence we have reviewed suggests that non-convexities exist; in fact, they are surprisingly common.

4. EFFICIENCY MEASUREMENT

In the previous two sections we discovered that convexity axioms exclude some interesting economic phenomena, and that numerous empirical studies suggest these phenomena are frequently observed in the real world. Still, convexity assumptions do play an important role in the literature of efficiency measurement. In the following, we explore the interrelationship between economic efficiency analysis and convexity assumptions. In addition, we investigate the role of convexity assumptions in establishing dual relationships between economic and technical efficiency measures. To keep discussion focused, we restrict attention to a single economic efficiency measure, the Farrell (1957) cost efficiency measure, and a single technical efficiency measure, the extended Farrell measure (see Russell, 1990), which is a simple modification of the traditional Farrell input measure.[5] These measures are frequently used in applied work, because many applications involve non-profit organizations that face exogenously fixed outputs but still operate in a competitive input market, e.g. universities or electricity companies. Still, the arguments directly carry over to alternative economic efficiency measures, like revenue efficiency and profit efficiency, and related technical efficiency measures.

Economic efficiency

The Farrell (1957) measure for cost efficiency compares actual cost to minimal cost for a given output level and input prices $w \in \Re_+^m$. For the input vector $x \in L(y)$, it is formally defined as:

$$(6) \qquad CE(x, w | L(y)) = \min_{x' \in L(y)} (x'w / xw).$$

[5] The same measure was introduced by Färe, Grosskopf, and Lovell. (1983, 1985) under the denomination "weak input measure of technical efficiency".

A standard result in duality theory of production is that convexity of input sets can be imposed without affecting the minimum cost level for a given price vector (see e.g. Diewert, 1978). Since cost is a linear and decreasing function of the inputs, potential non-convex parts of the input sets are irrelevant for determining the minimum cost level, and hence also for measuring cost efficiency. Formally, we find the following equality:

(7) $CE(x, w | L(y)) = CE(x, w | c(L(y)))$.

Figure 1 graphically illustrates this point in a two-input setting. The solid gray area $L(y)$ represents a hypothetical non-convex input set. The convex hull c($L(y)$) adds the striped area to this input set. Suppose we evaluate the cost efficiency of vector *A* at the relative prices reflected by the slope of the iso-cost line *l*. The vector CE minimizes cost over the true input set $L(y)$. That vector also minimizes cost over the convex hull c($L(y)$). Consequently, imposing convexity (in input space) does not affect the cost efficiency measure for vector *A*, which equals ($\frac{OCE'}{OA}$). It is easily verified that this result applies for all possible relative prices.

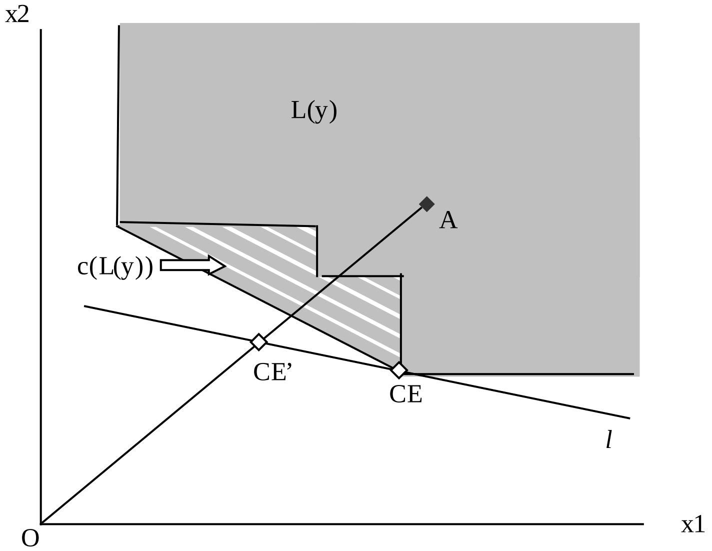

Figure 1: The harmless nature of convexity assumptions

Similarly, for analyzing revenue maximizing behavior, assuming convexity of the output sets is harmless, and for analyzing profit maximizing behavior, assuming convexity of the entire production set is harmless. We will not discuss this in detail here, since the analogy between the analysis of revenue maximization and the analysis of cost minimization is straightforward. For further discussion on convexity within the analysis of profit maximization, see e.g. Varian (1984) and Banker and Maindiratta (1988).

Interestingly, the DEA-related economic literature on non-parametric production analysis (see e.g. Afriat, 1972; Hanoch and Rothschild, 1972; and Varian, 1984) typically uses convexity only as an instrumental regularity property when it does not interfere with the assumed optimization hypotheses. That is, the convexity properties are motivated from the perspective of the economic objectives (such as cost minimization or profit maximization), not as inherent features of the technology. Similarly, the parametric approach (see e.g. Bauer, 1990) sometimes imposes regularity restrictions on the parameters of cost, revenue and profit functions, but it does not systematically impose convexity restrictions for the production function.

Finally, it is worth to stress that while convexifying the input sets is harmless for analyzing cost efficiency, it is not at all required for that undertaking. From (7), we see that the original input set $L(.)$ would yield exactly the same cost efficiency estimates as the convexified set $c(L(.))$

Technical efficiency
In practice, DEA is often used for analyzing technical efficiency rather than economic efficiency. For example, the prices for inputs and outputs frequently cannot be measured accurately enough to use economic efficiency concepts, e.g. because accounting data can give a poor approximation for economic prices (i.e. marginal opportunity costs). Several authors, including Charnes and Cooper (1985**,** cite this concern as a motivation for emphasizing technical efficiency measurement. Also, technical efficiency measures can test necessary optimality conditions for a wide range of firm objectives. For example, the extended Farrell technical input efficiency measure (introduced below) is consistent with all firm objectives that are decreasing functions of the inputs. Hence, it can also account for situations where DMU objectives are non-linear in inputs

The extended Farrell measure is defined as

(8) $$\theta(x \mid L(y)) = \min_{\theta}\left\{\theta \mid \theta x \in L(y) + \Re_{+}^{m}\right\}.$$

This measure equals the Farrell efficiency measure as computed with respect to monotonized input sets. Obviously, when $L(y)$ is truly monotone the Farrell measure and its extended counterpart coincide. In view of our discussion on monotonicity in the introductory section, the extended Farrell measure in fact incorporates inefficiency arising from (input) congestion as technical inefficiency (see Färe, Grosskopf, and Lovell, 1985, for a more elaborate discussion).

In contrast to cost efficiency analysis, 'convexification' of input sets is not a harmless undertaking when measuring technical efficiency. Obviously,

(9) $$\theta(x \mid L(y)) = \theta(x \mid c(L(y)))$$

holds as an identity only for convex input sets. That is, imposing convexity does make a difference when measuring technical efficiency, as the empirical results confirm (see Section 3). Therefore, when technical efficiency is the primary concern, the justification of convexity must come from the technology features.

Duality argument
Interestingly, cost efficiency measure (6) bears a direct relationship to technical efficiency, i.e. the extended Farrell measure (8) provides a natural upper bound for the cost efficiency measure. Formally,

$$(10) \qquad \theta(x|L(y)) \geq CE(x, w|L(y)) \quad \forall w \in \Re_{+}^{m}.$$

In words, if a DMU can reduce all inputs by at least a particular factor, then it can also reduce total cost by at least that factor. Hence, we may use the extended Farrell measure (which does not require price or cost information) as a 'proxy' for the cost efficiency measure (which does require such information).

This interrelationship between technical and economic efficiency measures is closely related to the duality relationships between production sets on the one hand and economic (profit, cost and revenue) functions on the other (see e.g. Shephard, 1953, 1970; McFadden, 1978; and Färe and Primont, 1995, for insightful treatments). It is well known that duality is intimately related to particular convexity properties. For example, the extended Farrell input efficiency gauged relative to the convex hull of the input set equals the cost efficiency measure evaluated at the "most favorable" prices, i.e.: [6]

$$(11) \qquad \theta(x|c(L(y))) = \max_{w \in \Re_{+}^{m}} CE(x, w|L(y)).$$

This means that 'convexification' (i.e. using $c(L)$ instead of L) can in fact improve the approximation of cost efficiency when price data is unavailable and we use the Farrell input efficiency measure as a substitute, i.e. $|\theta(y, x|c(L)) - CE(x, w|L(y))| \leq |\theta(y, x|L) - CE(x, w|L(y))|$.

Figure 2 continues the above example to illustrate this duality argument. The Farrell input efficiency value as computed relative to the convex hull $c(L(y))$ equals $\frac{ODF'}{OA}$, which gives a better upper bound for the cost efficiency value $\frac{OCE'}{OA}$ than the Farrell input efficiency value as obtained relative to the original non-convex input set $L(y)$, i.e. $\frac{ODF}{OA}$.

[6] This relationship follows as a straightforward corollary from the standard duality relationship between the input distance function and cost function (see e.g. Färe and Primont, 1995, pp. 48), recalling that Farrell input measure is the reciprocal of the input distance function, and cost efficiency is the ratio of cost function to the actualized cost. Equation (11) deviates from standard duality expressions in that it does not postulate L to be convex, but (for sake of generality) resorts to $c(L(.))$.

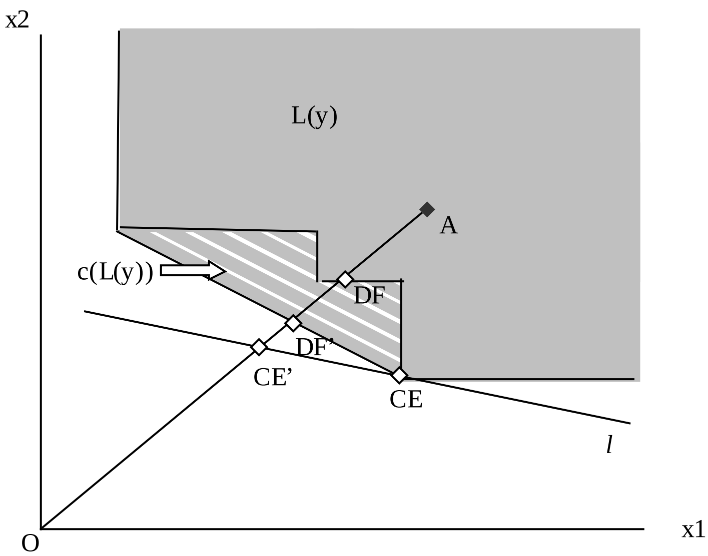

Figure 2: Duality argument for convexification

However, contrary to widespread opinion, we think this argument cannot satisfactorily justify the convexity axioms in the present context. The fact that the above duality result breaks down for non-convex technologies does not mean that economic efficiency measurement is not possible for such production sets. Rather, it means that we have to separate economic efficiency analysis from technical efficiency analysis. For example, we can measure cost efficiency as $CE(x, w|L(y))$ and technical efficiency as $\theta(x|L(y))$. In addition, an upper bound for cost efficiency can be directly gauged as $\max_{w \in \Re_+^m} CE(x, w|L(y))$ (without using $\theta(x|L(y))$ as a proxy). This upper bound equals $\max_{w \in \Re_+^m} CE(x, w|c(L(y)))$ obtained from the convexified input set. Therefore, the duality relationship does not add value to efficiency measurement.[7]

Two final qualifications are in order. Firstly, the above discussion makes clear that convexity assumptions are sometimes harmless (but not required!) for the analysis of economic efficiency. However, it is worth to emphasize that the convexity assumptions have to be in accordance with the employed efficiency measure. For example, cost efficiency allows for convex input sets, but not for convex output sets, let alone for a fully convex production set. In this respect, we do not know any economic efficiency measure that can motivate imposing convexity simultaneously for input sets and output sets, but not for the entire production set (as proposed by Petersen, 1990; and Bogetoft *et al.*, 2000). This is a valid concern, because the models proposed for imposing that assumption are computationally much more complex than the models that impose convexity in input space, output space, or the full input-output space (see Section 5).

Secondly, the above results are based on a series of simplifying assumptions. Specifically, the Farrell cost efficiency measure requires that prices are exogenously given (i.e. do not depend on output-input quantities), and are known with full

[7] Of course, duality is indispensable for recovering technology representations from cost, revenue, or profit functions, but that is another matter.

certainty by the DMUs. However, in many real-life applications, prices are not exogenous, but vary according to the actions by the DMU (see Chamberlin, 1933; and Robinson, 1933, for classic treatments). Furthermore, DMUs often face *ex ante* price uncertainty when making production decisions (see e.g. Sandmo, 1971, and McCall, 1967).[8] When proceeding towards more realism in these respects, the harmless character of convexity assumptions in efficiency analysis is further refuted (see e.g. Cherchye *et al.*, 2000). In general, for convexity to be a harmless production assumption, the economic objective function of the DMUs should be quasi-convex in the inputs and outputs. However, quasi-convexity excludes many common economic phenomena, such as risk aversion. In fact, quasi-concavity rather than quasi-convexity is the standard assumption for firm objectives in micro-economic theories of the firm.

5. Practical Considerations

In the previous section, we have focused on efficiency measurement when the true production set is fully known. However, the production set is generally unknown in practical applications, and the primary purpose of DEA is to construct an empirical production set from observations of a sample of *n* comparable DMUs. Below, we distinguish two categories of arguments that could motivate convexity assumptions in practical applications: *small sample performance* and *computational ease*.

Throughout this section, we will let $Y = (y_1 \cdots y_n)^T$, with $y_j = (y_{1j} \cdots y_{sj})$, j=1,…,n, denote the output vectors of the observations, and $X = (x_1 \cdots x_n)^T$, with $x_j = (x_{1j} \cdots x_{mj})$, j=1,…,n, the input vectors. In addition, we use the index sets $J = \{1,...,n\}$, and $I = \{1,...,m\}$.

Different DEA models build on different assumptions concerning the structure of the production technology and the quality of the data set. For sake of illustration, we compare two of the most popular models: the convexity-free model by Deprins *et al.* (1984) and the convex model by Banker *et al.* (1984).

The model by Deprins *et al.* (1984) uses the monotone hull of the observations as an empirical technology. Using the input set, this technology can be characterized as

(12) $$\hat{L}_{FDH}(y) = \{x | x \geq x_j ; y \leq y_j \quad j \in J\}.$$

The model by Banker *et al.* (1984) uses the convex monotone hull of the observations:

(13) $$\hat{L}_{BCC}(y) = \{x | x \geq \lambda X ; y \leq \lambda Y; \lambda e = 1; \lambda \in \Re^n_+\}.$$

Focusing on these two models entails a number of restrictions. First, Deprins *et al.* (1984) and Banker *et al.* (1984) assume that the output-input vectors are technically feasible, i.e. $x_j \in L(y_j), \ \forall j \in J$. This assumption is frequently debatable because data sets are often contaminated by errors-in-variables, e.g. because of the use of debatable valuation and depreciation schemes for accounting data. Second, both

[8] For efficiency analysis under ex ante price uncertainty, we refer to Kuosmanen and Post (1999).

models assume that the production set is monotone. In the introduction, we noted that congestion can violate monotonicity.[9] Third, the above two models differ only by the overall convexity assumption (A2), and hence we focus on the overall convexity axiom rather than relaxed convexity assumptions. Still, our arguments apply with equal strength to a wide range of alternative models that account for errors-in-variables[10], congestion, or entertain more relaxed convexity axioms.

SMALL SAMPLE PERFORMANCE

An important consideration for choosing between different approximations is the statistical goodness of the associated efficiency estimators. Obviously, this issue is irrelevant if the production assumptions in question are harmless (i.e. do not affect the estimates), e.g. the assumption of convex input sets for analyzing cost efficiency (see Section 4).

Under the maintained assumptions of feasibility and monotonicity (imposed by both models), the FDH approximation is contained in the true input set, i.e. $\hat{L}_{FDH}(y) \subseteq L(y)$, and hence the efficiency estimates $CE(x,w|\hat{L}_{FDH}(y))$ and $\theta(x|\hat{L}_{FDH}(y))$ bound the true efficiency measures $CE(x,w|L(y))$ and $\theta(x|L(y))$ from above. In addition, as the data set increases the approximation generally improves. In fact, the estimates are statistically consistent (i.e. they are (roughly speaking) asymptotically unbiased with a vanishing variance) for a wide range of statistical distributions.[11] Consistency also applies for the BCC approximation, but only if the true production set is convex. Hence, if the true production set is convex, FDH and BCC models generally yield approximately the same results in large data sets. However, if the production set is non-convex, the BCC set gives an inconsistent approximation. Therefore, in large-scale applications, convexity constitutes a source of specification error, but cannot improve the statistical goodness.

Nevertheless, the non-parametric efficiency estimators will usually converge slowly (especially when the number of input-output dimensions is high)[12], and may involve substantial small sample error even in quite extensive data sets. This is typically reflected in a relatively high number of self-identifiers, i.e. DMUs that are classified as efficient by lack of comparison (also labeled “efficient by default”). Imposing additional production structure can mitigate this problem, provided the assumptions are neither 'harmless', nor incorrect. For example, imposing convexity as in (13) can considerably increase the rate of convergence of the efficiency estimates and reduce small sample error when the convexity axiom (A2) truly holds. However, as argued in the previous sections, there are few empirical or theoretical arguments favoring a priori convexity axioms.

[9] Note that imposing monotonicity in input space does not interfere with the results for the extended Farrell technical efficiency measure as introduced in the previous section. Still, imposing monotonicity in output space can erroneously exclude congestion effects.

[10] See e.g. Post (1999, Section 4.2) for a detailed survey.

[11] For an elaborate treatment of the statistical properties of DEA estimators, we refer to Banker (1993), Korostolev *et al.* (1995), Park *et al.* (1997), Kneip *et al.* (1998), Gijbels *et al.* (1999), and Simar and Wilson (2000).

[12] This phenomenon is sometimes referred to as the “curse-of-dimensionality”, and it is relevant for most nonparametric regression techniques (see e.g. Yatchev, 1998).

Unfortunately, it is generally difficult to empirically test production hypotheses in DEA. The proposed specification tests (e.g. the *F*-tests introduced by Banker, 1993) invariably rely on comparing the efficiency estimates from a model that imposes the hypothesized production property with the estimates from a relaxed model that does not impose that property (e.g. comparing $\theta(x|\hat{L}_{FDH}(y))$ with $\theta(x|\hat{L}_{BCC}(y))$). In these tests, the production hypothesis is accepted if there is no statistically significant difference in the efficiency estimates of the two models. This approach requires large samples; in small samples it can confuse non-convexities with the small sample error associated with the relaxed model. However, as argued above, convexity cannot improve goodness in large samples. Hence, the specification tests can test production assumptions only under conditions in which those assumptions cannot improve the goodness of the estimators!

Due to the non-testable nature of production assumptions, we often face a difficult trade-off between small sample error (associated with the use of minimal assumptions), and specification error (associated with additional production assumptions). In this respect, "relaxed" convexity axioms could in some situations offer good compromise solutions. Still, we would emphasize that parsimony with assumptions is preferable in research situations for at least the following reasons:

1. Even if the statistical goodness is poor, the use of minimal production assumptions does preserve the logical property of the efficiency estimates as an upper bound to the true efficiency measures, and hence of full estimated efficiency as a necessary condition for full true efficiency. This is an important feature especially when type I errors in efficiency classification (i.e. diagnosing a 100% efficient DMU as inefficient) have more severe consequences than type II errors (diagnosing an inefficient DMU as efficient), e.g. when some form of reward or punishment is involved.

2. There are alternative strategies to correct for the small sample error without imposing additional structure on production possibilities. An obvious one is to invest further resources in expanding the sample by collecting additional data from existing DMUs or by introducing non-exiting 'virtual DMUs' constructed e.g. by consulting experts of the field in the spirit of Thanassoulis and Allen (1998). A more involved strategy is to correct for the small sample error using information of the asymptotic sampling distribution (see e.g. Park *et al.*, 1997) or a simulated sampling distribution generated by bootstrapping techniques (see e.g. Simar and Wilson, 1998). Finally, one can try to refine the efficiency measure to better reflect the underlying production objectives. For example, if input oriented technical efficiency measure is used as a proxy for cost efficiency, one could try to collect additional information on input prices to restrict the feasible price domain (see e.g. Kuosmanen and Post, 2001, for further discussion).

3. When only a small sample is available, parametric estimation techniques might become preferable over the nonparametric models, as the potential gain from introducing parametric structure (i.e. reducing the small sample error) becomes relatively large as compared to the potential specification error. In addition, detailed case studies that account for a wide range of DMU-specific information could become feasible if the number of DMUs is very small.

COMPUTATIONAL EASE

In practice, computational complexity can be an important consideration for choosing between different models. After all, a model that uses theoretically or empirically sound assumptions and gives statistically good estimates has little use if the computational burden prevents practical application.

The original DEA models, which rely on the assumption of convexity of the entire production set (Charnes *et al.*, 1978, Banker *et al.*, 1984, Charnes *et al.*, 1985) can be solved using simple linear programming[13]. For example, the cost efficiency estimate $CE(x,w|\hat{L}_{BCC}(y))$ can be computed by solving the linear programming problem

$$(14)\qquad \begin{aligned} &\min_{\lambda}\ \lambda Xw/xw \\ &s.t. \lambda Y \geq y \\ &\lambda e = 1 \\ &\lambda \geq 0 \end{aligned}$$

and the technical efficiency estimate $\theta(x|\hat{L}_{BCC}(y))$ can be computed by solving the linear programming problem

$$(15)\qquad \begin{aligned} &\min_{\lambda,\theta}\ \theta \\ &s.t. \lambda X \leq \theta x \\ &\lambda Y \geq y \\ &\lambda e = 1 \\ &\lambda \geq 0 \end{aligned} \quad .$$

This is an attractive feature, because linear programming is a standard mathematical programming technique, and LP solvers are included in many popular software packages, including spreadsheets.

However, despite the attractive LP structure, convex models need not be computationally more attractive than non-convex models. In fact, efficiency measures gauged relative to the monotone hull can be computed by using enumeration algorithms that involve even less computational burden than linear programming. For example, the cost efficiency estimate $CE(x,w|\hat{L}_{FDH}(y))$ can be computed by using the enumerative formulation

$$(16)\qquad \min_{j\in J: y_j \geq y} x_j w/xw,$$

[13] The same is true for the models that impose convexity in either input or output space, but not simultaneously in both.

and the technical efficiency estimate $\theta(x|\hat{L}_{FDH}(y))$ can be computed by using the enumerative formulation

(17) $$\min_{j\in J:y_j\geq y}\max_{i\in I}\left(x_{ij}/x_i\right).$$

See Tulkens (1993) and Cherchye *et al.* (2001b) for further details. In fact, the most efficient DEA computation routines for convex models employ the enumeration principle of convexity-free models to streamline computation (see Ali, 1993). See also Lovell and Vanden Eeckaut (1993) for more discussion on the computational efficiency of enumeration algorithms.

A special note applies to models that impose convexity in output space and input space but not in the full output-input space. Those models are computationally very complex. For example, the Bogetoft *et al.* (2000) solution algorithm relies on a recursive procedure for building the empirical production set as an ever larger union of convex sets. The computational burden for large-scale applications can be substantial. In fact, the number of recursions can become infinite, and it is not clear yet how to evaluate the approximation in case of premature termination. This is a rather discouraging feature, especially since the imposed convexity assumptions seem to have little economic meaning (as discussed in Section 3).

5. Summary and Discussion

Empirical, theoretical, and computational arguments favor dropping convexity axioms in DEA. Contrary to widespread opinion, micro-economic theory does not provide a valid argument for a priori convexification, even though convexity assumptions do play a central role in some theories of the firm. In some applications, it may be possible to back up convexity axioms by means of engineering process analysis. However, in many instances such convincing feedback is not available. In addition, empirical verification of convexity assumptions within the DEA methodology is problematic. In other traditions, a number of empirical studies suggest violations of convexity in a wide variety of industries. Furthermore, convexity-free efficiency estimates typically can be solved using simple enumeration algorithms, and adding convexity assumptions can substantially increase the computational burden.

The only valid argument we see for imposing additional convexity assumptions is the possible reduction of small sample error. However, that reduction comes at the cost of possible specification error (especially since empirical testing is problematic), and is likely to be negligible in large samples. In addition, the DEA methodology comprises alternative tools to reduce small sample error.

For these reasons, we call for a reorientation of future research towards convexity-free models. More generally, since alternative production assumptions (such as monotonicity and ray unboundedness) can be questioned as well, this calls for a reorientation of DEA research towards models that build more directly on the observed data set. Such a reorientation is in line with the economic literature on non-parametric production analysis, which in general does not impose any production axioms, unless they are harmless within that context. Such a reorientation also stresses the importance of the quality of the data set, and the analysis of errors-in-variables and outliers (see e.g. Post, 1999, Section 4.2 for a survey of techniques currently available in DEA). In addition, it calls for directing further research efforts towards the analysis of sampling error (see e.g. Grosskopf, 1996, for a survey of techniques available in the DEA methodology, and Simar and Wilson, 2000, for recent advances in that area), and the possible exploitation of additional production information (e.g. in the spirit of Thanassoulis and Allen, 1998).

In this paper, we focused on efficiency analysis, for which DEA was originally designed (see the seminal articles by Farrell, 1957, and Charnes *et al.*, 1978). We have abstracted from the study of production frontier characteristics such as returns-to-scale properties and substitution elasticities, which might be obtained as 'side-products' of DEA efficiency analysis. Still, many of the above arguments (empirical evidence and practical considerations) apply with equal strength to these characteristics. However, the duality arguments in favor of convexity for the purpose of efficiency measurement do not apply.

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