DEA WITH EFFICIENCY CLASSIFICATION PRESERVING CONDITIONAL CONVEXITY

by

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Abstract: We propose to relax the standard convexity property used in Data Envelopment Analysis (DEA) by imposing additional qualifications for feasibility of convex combinations. We specifically focus on a condition that preserves the Koopmans efficiency classification. This yields an efficiency classification preserving conditional convexity property, which is implied by both monotonicity and convexity, but not conversely. Substituting convexity by conditional convexity, we construct various empirical DEA approximations as the minimal sets that contain all DMUs and are consistent with the imposed production assumptions. Imposing an additional disjunctive constraint to standard convex DEA formulations can enforce conditional convexity. Computation of efficiency measures relative to conditionally convex production set can be performed through Disjunctive Programming.

Key words: *Data envelopment analysis, Nonparametric efficiency analysis, Conditional convexity, Disjunctive Programming*

1. INTRODUCTION

Data Envelopment Analysis (DEA), originating from Farrell's (1957) seminal work and popularized by Charnes, Cooper and Rhodes (1978), provides a flexible nonparametric doctrine for empirical production analysis. In recent decades, DEA has rapidly expanded towards new application areas (see e.g. Seiford (1996) for a survey). In addition to its original use in efficiency measurement, DEA is also employed for approximating production possibility sets or input/output correspondences (see Färe and Grosskopf (1995)), recovering shadow prices (see Färe, Grosskopf and Nelson (1990)), providing best-practice benchmarks (e.g. Bogetoft and Hougaard (1999)), as monitoring tool in agency problems (Bogetoft (1994)), and as a "lazy man's decision support tool" (Doyle (1995)).

The key contribution of DEA to efficiency analysis, and empirical production analysis in general, is the possibility to approximate unobservable production technologies from empirical input-output data of Decision Making Units (henceforth DMUs) without imposing overly restrictive parametric assumptions. See e.g. Bauer (1990) for a discussion of parametric frontier models. Still, DEA models usually impose *monotonicity* (i.e. free disposability of *all* inputs and outputs) and *convexity* assumptions, sometimes complemented with assumptions concerning returns to scale properties (see e.g. Seiford and Thrall, 1990).

It is to be noted that the basic determinism postulate of DEA, which states the production set should contain all observed Decision Making Units (DMUs), already suffices for many important functions of DEA. Additional properties such as monotonicity and convexity become useful when we wish to extend the scope of the analysis to assessments concerning the degree of inefficiency (using e.g. the radial Debreu-Farrell measures), and the shape of the "best-practice" production frontier. Unfortunately, these assumptions can be viewed overly restrictive, too. In fact, these properties can be violated in many important situations well recognized in economic theory. *Congestion* of production factors (see Färe and Svensson, 1980) can violate monotonicity. Many other features such as increasing marginal product of inputs, or indivisibility of inputs and outputs can violate convexity. To quote McFadden (1978, p.

8): "*Convexity… holds if the technology is such that substitution of one input combination for a second, keeping output constant, results in a diminishing marginal reduction in the second input combination, or if production activities can be operated side by side (or sequentially) without interfering with each other. However, the importance of* [monotonicity and convexity] *in traditional production analysis lies in their analytic convenience rather than in their economic realism.*"

Restrictiveness of the standard monotonicity and convexity conditions provides a substantial motivation to look for more general properties, on which DEA models could be more firmly based. In this paper we propose to relax the convexity property by imposing additional qualifications on technical feasibility of convex combinations of observed DMUs. The focus on convexity instead of monotonicity is justified by the fact that unlike monotonicity, convexity does not interfere with the static taxonomy of efficiency measures presented by Farrell (1957) and extended by Färe, Grosskopf and Lovell (1983, 1985). We specifically focus on developing a condition, which preserves the standard efficiency classification based on the classic Koopmans (1951) definition. This condition yields a very general conditional convexity property, which is implied by both monotonicity and convexity, but not conversely. Substituting convexity by conditional convexity, we construct various empirical DEA approximations as the minimal sets that contain all DMUs and are consistent with the imposed production assumptions. Imposing an additional disjunctive constraint to the convex DEA models can enforce conditional convexity. Computation of efficiency measures relative to this production set can be performed through Disjunctive Programming.

The rest of the paper unfolds as follows. Section 2 presents the necessary notation and terminology by reviewing the static taxonomy of efficiency measures. Section 3 discusses the role of convexity and monotonicity properties in the DEA reference technologies. Section 4 presents the central concept of this paper: viz. conditional convexity. In Section 4 we also discuss how an empirical production set can be constructed based on efficiency preserving conditional convexity. In Section 5 we formulate the Disjunctive Programming problems for characterizing various reference technologies and computing efficiency measures relative to them. Section 6 illustrates the approach by a simple numerical example. Finally, Section 7 draws our conclusive remarks.

2. STATIC TAXONOMY OF EFFICIENCY MEASURES

DEA models assess various performance dimensions of decision-making units (DMUs) that allocate inputs $x = (x_1 \dots x_q)^T \in \Re_+^q$ to produce outputs $y = (y_1 \dots y_p)^T \in \Re_+^p$. Inputs and outputs are assumed to be observable and to completely characterize the production process. In this paper we focus on the static or cross-sectional efficiency analysis. For dynamic performance evaluation, see e.g. Färe and Grosskopf (1996). This section presents the necessary conceptual apparatus for the subsequent sections by discussing the static taxonomy of technical efficiency. For sake of simplicity, we mostly focus on input side, but the framework laid down in this section extends to output and full input-output space as well, see e.g. Färe, Grosskopf and Lovell (1985) for further discussion.

In this paper we characterize the production technology in terms of the *production possibility set*

$$T = \{(x, y) \in \Re_+^{q+p} \mid \text{input } x \text{ can produce output } y\}.$$

The production set T is generally assumed to be closed and nonempty. Additional properties discussed in this paper include:

Monotonicity: *A production set T is said to be monotonous if for all* (*x*,*y*): $(x, y) \in T \Rightarrow (x+u, y-v) \in T \ \forall u \in \Re_+^q, v \in \Re_+^p$.

Convexity: *Let X and Y denote a* $(q \times n)$ *input matrix and a* $(p \times n)$ *output matrix respectively, and let* X_j (Y_j) *denote the column j of X* (*Y*). *A production set T is said to be convex if for all X,Y:* $(X_j, Y_j) \in T \ \forall j = 1,...,n \Rightarrow (Xl, Yl) \in T \ \forall\, l \in \Re_+^n, el = 1$.

Constant Returns to Scale (CRS): *A production set T is said to exhibit constant returns to scale if* $T = aT, a > 0$.

Four remarks of these properties are worth noting. First, monotonicity is equivalent to free disposability of *all* inputs and outputs. We think of monotonicity as a property of a production set, while disposability is a property associated with inputs and outputs. Second, monotonicity and convexity properties can be straightforwardly associated with subsets of T (i.e. input or output correspondences) as well. Monotonicity (convexity) of production set naturally implies monotonicity (convexity) of these subsets, but the converse need not hold. Third, returns to scale properties are commonly associated with a production set, but also with production vectors, as discussed below in more detail. Finally, the basic DEA toolbox includes alternative returns to scale properties as well, see e.g. Seiford and Thrall (1990). In this paper we mainly focus on the constant returns to scale (CRS) property. We briefly note in Section 5 that the basic alternative properties are obtained by simple modifications to the models presented here.

In what follows we shall also employ an equivalent representation of production possibilities: the *input distance function* by Shephard (1953) defined as

$$D_T(x,y) = Sup\{q \in \Re_+ \,|\, (x/q, y) \in T\}.$$

If inputs are weakly disposable, i.e. $(x,y) \in T \Rightarrow (x/q, y) \in T \forall q \in (0,1]$, then $D_T(x, y) \geq 1$ is equivalent to $(x, y) \in T$, and $D_T(x, y) = 1$ characterizes the *isoquant* of the input correspondence (e.g. Färe (1988)).

Following Koopmans (1951) (see also Färe, 1988), the efficient subset of the production set T can be defined as

$$Eff.T = \{(x,y) \in T \,|\, x' \leq x, y' \geq y, (x', y') \neq (x,y) \Rightarrow (x', y') \notin T\}.$$

In the subsequent sections we will also employ another relevant subset, the weak efficient subset defined as (e.g. Färe, 1988):

$$WEff.T = \{(x,y) \in T \,|\, x' < x, y' > y \Rightarrow (x', y') \notin T\}.$$

Clearly, $WEff.T \supseteq Eff.T$. That is, efficiency implies weak efficiency, but the reverse relationship should not necessarily hold.

There are a number of alternative measures for gauging the degree of inefficiency (see e.g. De Borger et al. (1998)). In this paper we confine attention to the standard Debreu-Farrell input measure, which is simply the inverse of the input distance function, i.e.

$$DF_T(x, y) = D_T(x, y)^{-1} . \qquad (1)$$

As noted by Färe and Lovell (1978), the Debreu-Farrell input measure can fail to fully account for all inefficiency in the sense of Koopmans. Most notably, the Debreu-Farrell measure may equal unity for an inefficient DMU, and the reference point $(DF_L(x, y)x, y)$ does not necessarily belong to the efficient subset. Nevertheless, the remaining nonradial 'slacks' are in the original Farrell's framework captured by the allocative efficiency component, which depends on the objective function of the producer. In this paper we leave the explicit producer's objective unspecified, and hence focus on technical efficiency.

Färe, Grosskopf and Lovell (1983, 1985) extended the static taxonomy to include additional notions of *structural efficiency* and *scale efficiency*. For these purposes, define the *monotone hull* (*MH*) of *T* as the smallest monotone set that contains *T*, i.e.

$$MH(T) = \left\{ (x, y) \middle| x = x' + u; y = y' - v; (x', y') \in T; (u, v) \in \Re_+^{q+p} \right\}.$$

Next, define the smallest monotone set that exhibits CRS and contains *T* as the *ray-unbounded monotone hull* (RMH), i.e.

$$RMH(T) = \left\{ (x, y) \middle| (x, y) \in \ l MH(T),\ l > 0 \right\}.$$

Structural efficiency (STR) and scale efficiency (SCA) can now be defined as

$$STR_T(x, y) = \frac{DF_{MH(T)}(x, y)}{DF_T(x, y)} \text{ and}$$

$$SCA_T(x,y) = \frac{DF_{RMH(T)}(x,y)}{DF_{MH(T)}(x,y)}$$

respectively. Production is diagnosed structurally efficient if and only if it occurs in a non-congested or "economic" region of production, where marginal product of every input is non-negative. Moreover, production is diagnosed scale efficient if and only if production takes place at *the most productive scale size* (see Banker et al. (1984) for discussion). Note that producing on the most productive scale need not always comply with the primary objectives of the producer (e.g. profit maximization, cost minimization), and consequently, scale efficiency should not be automatically included in overall efficiency criteria. For further details of structural and scale efficiency measures, see Färe, Grosskopf and Lovell (1983, 1985).

3. DEA APPROACH

The previous section phrased in terms of a theoretical production set *T*. Too often, the true production technology cannot be characterized precisely enough by engineering blueprints or other theoretical knowledge of the production process. The key feature of DEA is that it allows us to approximating the production technology directly from the observed production data. Let our observed sample consist of *n* DMUs, and denote the matrix of output vectors by $Y = \left(y^1 \cdots y^n\right)$ and the matrix of input vectors by $X = \left(x^1 \cdots x^n\right)$. We use $l = \left(l^1 \cdots l^n\right)^T$ to denote the vector of intensity variables, and $e = (1 \cdots 1)$. Finally, $S = \{1,...,n\}$ denotes an index set of cardinality *n*.

In deterministic DEA it is generally assumed that all observed production vectors are technically feasible, i.e. $(X_j, Y_j) \in T \ \forall j \in S$. This seems reasonable if the relevant input-output variables can be measured accurately enough; after all, these input-output combinations are observed. As already suggested by Tulkens and Vanden Eeckaut (1999), this basic assumption already allows one to distinguish between efficient, weak efficient, and inefficient DMUs respectively, i.e.

$$Eff.(X,Y) = \left\{ j \in S \,\middle|\, \nexists i \in S : x_i \leq x_j, y_i \geq y_j, (x_i, y_i) \neq (x_j, y_j) \right\}, \qquad (2)$$

$$WEff.(X,Y) = \{ j \in S \mid \nexists i \in S : x_i < x_j, y_i > y_j \}, \quad (3)$$

$$Ineff.(X,Y) = \{ j \in S \mid j \notin WEff.(X,Y) \}. \quad (4)$$

Distinguishing these subsets does not require any assumptions other than the determinism condition discussed above. Consequently, we will refer to these subsets as the 'spontaneous' efficiency classification. These sets are sufficient as such e.g. for benchmarking purposes (see Bogetoft and Hougaard (1999)). Also ordinal ranking of DMUs based on the number of dominating/dominated DMUs is possible (Tulkens and Vanden Eeckaut (1999)).

Further assumptions are typically necessary for measuring the (cardinal) degree of inefficiency in terms of the standard taxonomy. The original DEA models formulated by Farrell (1957) and Charnes et al. (1978) were based on the maintained assumption that the production set T satisfies all three additional properties given in Section 2: monotonicity, convexity, and CRS. This assumption allows us to approximate the production set by the *ray-unbounded convex monotone hull* (henceforth *RCMH*) of the observed production vectors, i.e.

$$RCMH(X,Y) = \left\{ (x,y) \in \Re_+^{q+p} \middle| \begin{pmatrix} y \\ -x \end{pmatrix} \leq \begin{pmatrix} Yl \\ -Xl \end{pmatrix}; l \in \Re_+^n \right\}.$$

Consistent with the *minimum extrapolation principle* (see Banker et al. (1984) and Bogetoft (1994)), *RCMH*(*X*,*Y*) is the minimal set that contains all observations and complies with the imposed properties. Although *RCMH*(*X*,*Y*) does not impose any parametric structure on production possibilities, postulating T to be monotonous and convex and to exhibit CRS is quite restrictive. Nevertheless, for all convex technologies, *RCMH*(*X*,*Y*) is contained in the congestion and scale adjusted reference set *RMH*(*T*) used in the static decomposition.

In the last two decades, a considerable development towards more general properties has taken place in DEA. Perhaps the most standard DEA reference technology, which relaxes the assumption of CRS but maintains assumptions of monotonicity and convexity, is the *convex monotone hull* (henceforth *CMH*) (see Afriat (1972), Färe, Grosskopf, and Logan (1983), and Banker et al. (1984)) defined as

$$CMH(X,Y)=\left\{(x,y)\in\Re_{+}^{q+p}\left|\begin{pmatrix}y\\-x\end{pmatrix}\leq\begin{pmatrix}Yl\\-Xl\end{pmatrix};el=1;\,l\in\Re_{+}^{n}\right.\right\}.$$

Obviously, $CMH(X,Y)\subseteq RCMH(X,Y)\ \forall(X,Y)$. Moreover, for convex *T*, *CMH*(*X,Y*) is contained within *MH*(*T*). Hence, *CMH*(*X,Y*) can be viewed as a proxy for the congestion adjusted reference set *MH*(*T*) in the static taxonomy.

We can proceed by relaxing monotonicity from CMH, which yields the *convex hull* (*CH*) (Charnes et al. (1985)), i.e.

$$CH(X,Y)=\left\{(x,y)\in\Re_{+}^{q+p}\left|\begin{pmatrix}y\\-x\end{pmatrix}=\begin{pmatrix}Yl\\-Xl\end{pmatrix};el=1;\,l\in\Re_{+}^{n}\right.\right\}.$$

Now, $CH(X,Y)\subseteq CMH(X,Y)\ \forall(X,Y)$. Moreover, for convex *T*, *CH*(*X,Y*) is contained in *T*, which justifies its use as a reference technology. Unfortunately, the assumption of convex *T* is almost void of economic realism. Like noted by McFadden (1978), it is mainly used for its analytical convenience. See Guesnerie (1975) and Cherchye et al. (2000), for further discussion.

In application oriented DEA, empirical realism should outweigh analytical convenience. Perhaps the most important drawback of this property lies in the fact that convexity typically interferes with the spontaneous efficiency classification. Hence, an erroneously imposed convexity assumption can have a substantial impact on policy recommendations. Dramatic effects of the convexity postulate have been reported in empirical studies. See e.g. Deprins et al. (1984), Tulkens (1993), Dekker and Post (1999), and Kuosmanen (1999). For example, in the retail banking application reported by Tulkens (1993, p. 192-197), 74.6% of public bank branches were found efficient assuming monotonicity only, but only 5.2% of all bank branches remained efficient when both monotonicity and convexity were imposed. For private bank branches, the same study reports 57.8% average efficiency without convexity, and 5.5% efficiency with convexity. We do not know which of these results are more correct, but it obviously takes a lot of faith to maintain convexity hypothesis without any suspicion.

Consequently, an alternative strategy has been considered in the literature, see e.g. Deprins et al. (1984) and Tulkens (1993). In this approach it is the convexity property that is relaxed, and monotonicity that is maintained. This gives the so-called Free Disposable Hull (FDH), which we here dub for sake of uniformity as *monotone hull* (henceforth *MH*), i.e.

$$MH(X,Y) = \left\{ (x,y) \in \Re_+^{q+p} \left| \begin{pmatrix} y \\ -x \end{pmatrix} \leq \begin{pmatrix} Yl \\ -Xl \end{pmatrix} ; el = 1;\ l^j \in \{0,1\};\ j \in S \right. \right\}.$$

A particularly attractive feature of monotonicity is its redundancy in the standard Koopmans notion of technical efficiency. That is, in contrast to convexity, monotonicity does not interfere with the efficient, weak efficient, or inefficient subsets (2)-(4).

If *T* is monotonous, it contains *MH*(*X,Y*). Nevertheless, monotonicity of *T* can be viewed equally unrealistic as convexity. See e.g. Färe and Svensson (1980) for discussion on congestion. According to Färe, Grosskopf and Lovell (1983), examples of efficiency losses due to congestion include traffic congestion in the production of transportation, reduced grain yield due to excessive fertilization in agriculture, and output losses due to featherbedding and other union work rules.

From the applied point of view, *MH* has also other serious limitations. One frequently cited shortcoming is that after accounting for potential radial inefficiency measured by the Debreu-Farrell measure (1), considerable non-radial slacks typically remain (see e.g. De Borger et al. (1998) for discussion). Secondly, the marginal properties of *MH* are rather bizarre: Consider for example the *scale elasticity* measure that can be defined in terms of the distance function and its gradient (see e.g. Färe, Gosskopf, and Lovell (1988)) as

$$e(x,y) = -\frac{D_T(x,y)}{\nabla D_T(x,y)\cdot(0,y)}. \qquad (5)$$

Scale elasticity is an important indicator of returns to scale: increasing, constant or decreasing returns to scale are said to prevail when $e > 1$, $e = 1$ or $e < 1$. In case of *MH*, however, practically useful approximation of scale elasticity cannot be obtained due to the discrete nature of *MH*. The same problem equally concerns the elasticities of substitution and transformation. Thirdly, *MH* is subject to a considerable small sample error, which raises a need for large data sets. Although statistical procedures to deal with the third limitation have been developed (see e.g. Park et al. (1997), Gijbels et al. (1998), Simar and Wilson (1998)), at least the first two shortcomings remain.

Finally, a number of intermediate models that typically intend to weaken the general convexity assumption underlying *CMH*(*X*,*Y*) have been presented. In the literature on nonparametric production analysis, Hanoch and Rothschild (1972) and Varian (1984) have discussed the possibility of imposing convexity on the input correspondence, which does not necessarily imply convexity of the production set. In the DEA literature, Petersen (1990), Bogetoft (1996), Bogetoft et al. (forthcoming), and Post (forthcoming) have further developed the idea of convex input/output correspondences of a nonconvex production set. Furthermore, Kerstens and Vanden Eeckaut (1998, 1999) have presented models that impose monotonicity and CRS, but not convexity.

Consider, for example, a production set that starts from *MH* and additionally imposes convexity of input correspondences, i.e.

$$CIMH(X,Y) = \left\{ (x,y) \in \Re_{+}^{q+p} \middle| x \geq Xl\,;\, l^{j} y \leq l^{j} y^{j} \ \forall j \in S; el = 1; l \in \Re_{+}^{n} \right\} \quad .$$

There are some special occasions where convexity of input correspondence can be a reasonable and harmless requirement, e.g. when the managerial objective function is linear in inputs or outputs. See for example Hanoch and Rothschild (1972) or Kuosmanen and Post (1999a) for discussion on the economic interpretation of *CIMH*(*X*,*Y*) in terms of cost efficiency. Unfortunately, the general justification of these partially convex models seems problematic, especially as the analysis of pure technical efficiency is concerned. These models clearly combine the negative features such as ignorance of congestion and ad hoc specification of convexity, but the concrete benefits of these alternatives are unclear.

Yet, it has been suggested that convexity of input and/or output correspondences would be a standard property of technology. Specifically, Petersen (1990, p. 307) refers to the microeconomic "*law of diminishing marginal rates of substitutions*" that would imply such property. Also Bogetoft (1996) and Bogetoft et al. (forthcoming) cite this law as the motivation for their production assumptions. However, to the best of our knowledge, such law does not exist in economic theory. There are such classic laws of *diminishing returns* and *variable proportions* (see e.g. Schumpeter, 1966), but (as pointed out by both referees) these laws are related to monotonicity or lack of it (see Färe and Svensson, 1980) rather than convexity.

To demonstrate the difficulties with a maintained hypothesis of convex input correspondence, consider the following simple exercise in differential calculus. As noted above, the production frontier can be represented using the input distance function, i.e. $D_T(x,y) = 1$, akin to a *production function* in an implicit form. For analytical convenience, assume D_T is twice continuously differentiable. The following general expression can be derived by differentiating the implicit production relation $D_T = 1$ with respect to inputs i and j:

$$\left.\frac{dx_i}{dx_j}\right|_{D(x,y)=1} = -\frac{\partial D_T(x,y)/\partial x_j}{\partial D_T(x,y)/\partial x_i}. \tag{6}$$

This expression defines the marginal rate of substitution between inputs i and j in the neighborhood of the frontier point (x, y), i.e. the slope of the input isoquant. When (x, y) is non-congested, the distance function increases in x, and the partial derivatives in (6) are positive, which gives the usual decreasing input isoquants. Differentiating (6) again with respect to x_j (omitting the arguments (x,y) in D_T) gives

$$\left.\frac{d^2x_i}{dx_j^2}\right|_{D(x,y)=1} = \frac{\left(\partial^2 D_T/\partial x_i\,\partial x_j\right)\cdot\left(\partial D_T/\partial x_j\right) - \left(\partial^2 D_T/\partial x_j^2\right)\cdot\left(\partial D_T/\partial x_i\right)}{\left(\partial^2 D_T/\partial x_i^2\right)^2}.$$

This second derivative reflects the curvature of the isoquant. Convexity of the input isoquant requires nonnegative second derivative for all x. Clearly, the sign of the nominator is determinant in this respect.

Now what happens in case of increasing returns if the marginal product of input *j* is increasing, i.e. $\partial^2 D_T / \partial x_j^2 > 0$? Consider a noncongested point (*x*,*y*) where the both first-order partial derivatives are positive. Convexity of the input isoquant is preserved if

$$\partial^2 D_T / \partial x_i \partial x_j \geq \frac{\partial D_T / \partial x_i}{\partial D_T / \partial x_j} \partial^2 D_T / \partial x_j^2 .$$

In case the marginal product is increasing, the right hand side of this inequality is strictly positive. The left-hand side of the inequality represents the pure substitution effect in production, which could be positive or negative depending on the nature of inputs. Inputs *i* and *j* can be called technical complements *if* $\partial^2 D_T / \partial x_i \partial x_j \geq 0$, and technical substitutes if $\partial^2 D_T / \partial x_i \partial x_j \leq 0$. Clearly, *all* inputs must be sufficiently complementary in order to preserve convexity of the input isoquant.

In verbal terms, increasing marginal products can provide substantial incentives for utilizing scale economies by specialization. In case of substitute inputs, the specialization and substitution effects are mutually enhancing, which results as a concave input isoquant (non-convex input correspondence). In case of complementary inputs, however, there is a conflict between substitution and specialization effects. When convexity of the whole production set is relaxed, maintaining convexity of the input correspondence requires extreme complementarity in all inputs that always outweighs any scale effect. This seems a difficult assumption, as the relative magnitudes of the scale and substitution effects are difficult (if not impossible) to ascertain a priori.

4. CONDITIONAL CONVEXITY

As sufficiently general empirical production sets apparently cannot be built on the standard monotonicity and convexity properties, it is worth to consider alternative weaker technology properties. Since both monotonicity and CRS properties have already been given an explicit interpretation in terms of the static taxonomy, it seems fruitful to focus on the remaining convexity property. This section proposes a relaxation of the convexity property, which we call *conditional convexity*.

Definition *(General conditional convexity)**:** Let C denote an arbitrary well-defined logical condition. Production set T is convex conditional upon C, if for all X,Y* and $l \in \Re_{+}^{n}, el = 1: (X,Y) \in T \wedge C \Rightarrow (Xl, Yl) \in T$.

Clearly, if production set is convex, it is also conditionally convex for any well-defined *C*, but the converse need not be true.

To operationalize the general conditional convexity property, we need to specify the condition *C* explicitly. One particularly interesting possibility is to enforce the spontaneous technical efficiency classification by putting convexity conditional upon its preservation. This condition is motivated by the argument that the appeal of the frequently imposed monotonicity property mostly underlies in the similar feature, rather than in its economic realism. Hence, in this paper we focus on exploring convexity conditional specifically upon preservation of efficiency classification. In terms of Definition 4.1, the condition C is defined as

Definition (*Condition* C^{EP}): $Xl \not< x, Yl \not> y \;\; \forall (x,y) \in WEff.T$

Substituting *C* by Condition C^{EP} in the definition of general conditional convexity gives the *efficiency classification preserving conditional convexity* property. For sake of brevity, we will henceforth refer to this property by the abbreviated term *c-convexity*. Specifically, this property states that if no weak efficient production plan is dominated by a convex combination of feasible production plans contained in the production set, then the convex combination is feasible.

As noted above, convexity implies c-convexity, but the converse need not be true. Interestingly, the same holds for monotonicity, as the following proposition demonstrates:

PROPOSITION: If $T \subset \Re_{+}^{p+q}$ represents a closed, nonempty, monotone production set, then *T* is c-convex.

Proof: Let *X'* and *Y'* denote input and output matrices of an arbitrary subset of *m* production vectors $(X'_j, Y'_j) \in T \; \forall j = 1,...,m$, and let $(x(l), y(l))$ denote an arbitrary convex combination $x(l) = X' l; y(l) = Y' l; el = 1; l \in \Re^n_+$. The production set T is c-convex if for all $l, (x, y) \in WEff.T$: $x(l) \not< x, y(l) \not> y \Rightarrow (x(l), y(l)) \in T$. By monotonicity of T and the definition of the weak efficient subset, $(\tilde{x}, \tilde{y}) \notin T$ is equivalent to $\exists (x, y) \in WEff.T : \tilde{x} < x, \tilde{y} > y$. That is, an infeasible production vector necessarily dominates at least one weak efficient production vector. If this is not true for $(x(l), y(l))$, it immediately follows that $(x(l), y(l)) \in T$. *Q.E.D.*

As c-convexity does not generally imply monotonicity (see e.g. Figure 1), c-convexity can be viewed by the previous proposition as a more general property of production sets than monotonicity or convexity.

Now, how could we use c-convexity in DEA? In fact, empirical DEA production sets can be constructed on c-convexity in a similar fashion as on standard convexity. Note that the c-convexity depends on the weak efficient subset of technology. As the theoretical weak efficient subset is unknown, we resort to its empirical approximation *WEff.*(*X*, *Y*). The minimal set containing all DMUs and satisfying c-convexity is the *c-convex hull* (*CCH*) defined as

$$CCH(X,Y) = \left\{ (x, y) \in \Re^{q+p}_+ \middle| \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} Yl \\ -Xl \end{pmatrix}; el = 1; l \in \Re^n_+; C^{EP} \right\}. \quad (7)$$

Imposing monotonicity in addition to c-convexity gives the *c-convex monotone hull* (*CCMH*)

$$CCH(X,Y) = \left\{ (x, y) \in \Re^{q+p}_+ \middle| \begin{pmatrix} y \\ -x \end{pmatrix} \leq \begin{pmatrix} Yl \\ -Xl \end{pmatrix}; el = 1; l \in \Re^n_+; C^{EP} \right\}. \quad (8)$$

Finally, imposing CRS in addition to c-convexity and monotonicity yields the *ray-unbounded c-convex monotone hull* (*RCCMH*), i.e.

$$RCCMH(X,Y) = \left\{ (x, y) \in \Re^{q+p}_+ \middle| \begin{pmatrix} y \\ -x \end{pmatrix} \leq a \begin{pmatrix} Yl \\ -Xl \end{pmatrix}; a > 0; el = 1; l \in \Re^n_+; C^{EP} \right\}. \quad (9)$$

Figure 1 illustrates *CCH*, *CCMH* and *RCCMH* sets in a single input-output case. Black and white dots represent DMUs, the black ones are weak efficient, the white ones are inefficient. The dashed line with short gaps represents the boundary of *CCH*, the solid line represents the boundary of *CCMH*, and the straight line with long dash represents the *RCCMH* frontier.

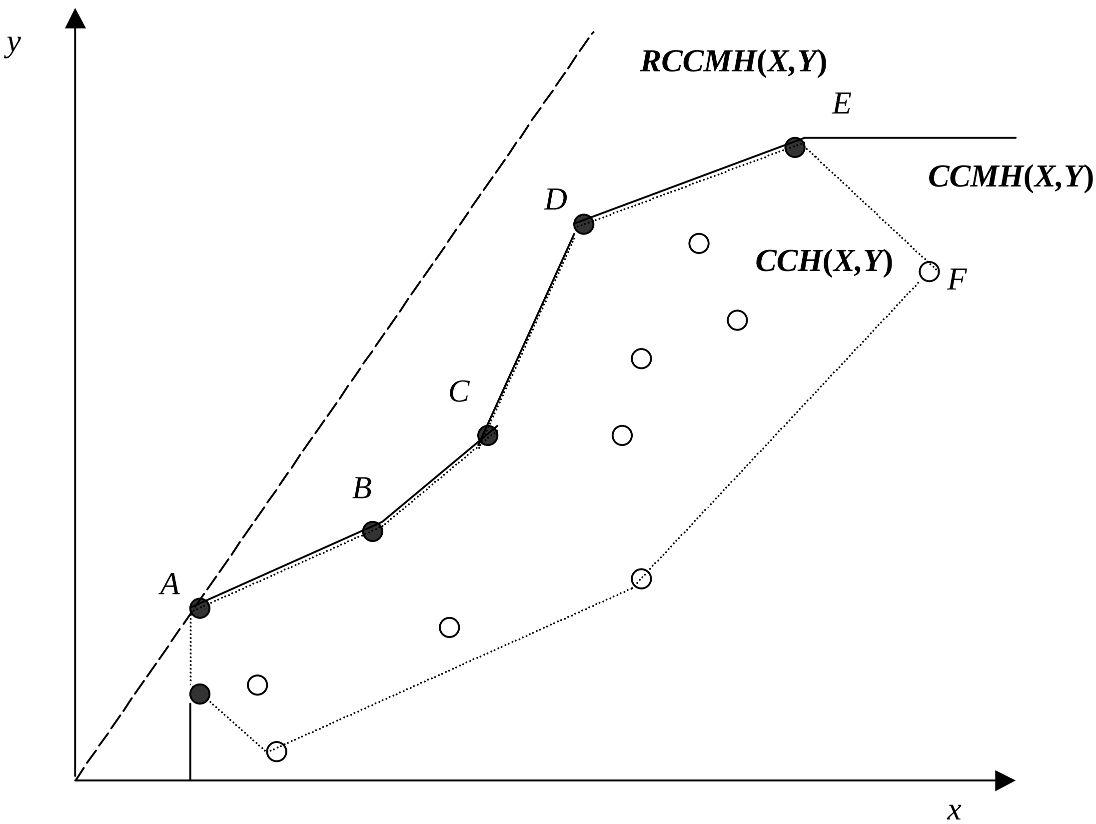

Figure 1: Illustration of CCH, CCMH and RCCMH sets

Three points are worth noting: First, both c-convexity and monotonicity preserve efficiency status of all weak efficient DMUs, but that feature need not carry over to the CRS property, like Figure 1 illustrates. Note that preserving efficiency classification under CRS is unnecessary. Note also that *RCCMH* differs from the CRS model by Charnes et al. (1978) in that it need not be convex in input and output spaces. Second, *CCH* and *CCMH* production sets do not restrict returns to scale to behave monotonically as *CMH* does: From A to C via B these technologies exhibit decreasing returns to scale, from C to D increasing returns prevail, while from D to E again decreasing returns prevail. In *CMH* there can only be a single increasing part followed by a decreasing part, which is a strong restriction. Third, *CCH* set indicates DMU *F* could suffer from congestion, which we would overlook by resorting to *MH* or other monotone approximations. However, one should take the inherent small sample error into account before drawing conclusions based on these methods.

Consider properties of these c-convex approximations. Note first that unlike monotonicity or convexity, c-convexity of *T* does not guarantee *CCH*(*X*,*Y*) would be contained within *T*. It is somewhat unfortunate that the attractive general character of the c-convexity property does not fully carry over to finite sample approximations. Still, c-convexity remains a more general property than convexity even in small samples. Clearly, all c-convex empirical sets are contained within their corresponding convex counterparts, i.e. $CCH(X,Y) \subseteq CH(X,Y)$, $CCMH(X,Y) \subseteq CMH(X,Y)$, and $RCCMH(X,Y) \subseteq RCMH(X,Y)$ for all (*X*,*Y*). This further implies that the c-convex approximations yield higher efficiency measures than their convex counterparts, e.g. $DF_{CCMH}(x,y) \geq DF_{CMH}(x,y)$ for all (*x*,*y*). Naturally, the c-convex empirical sets coincide with their convex counterparts if no convex combination interferes with the efficiency classification.

As for monotonicity, *CCH* need not contain *MH*. Conversely, *MH* need not contain *CCH* either. Consequently, although c-convexity is a more general property of theoretical production sets than monotonicity, this need not be the case in empirical approximations. Nevertheless, as *CCMH* is monotonous, it necessarily contains *MH*. Note that although monotonicity of *T* implies conditional convexity, *MH* need not contain *CCMH*, as a set of discrete DMUs presented by (*X*,*Y*) does not satisfy monotonicity. Still, we conjecture that if *MH* converges to a true monotone production set *T* as the sample size increases (see Park et al. (1997) for sufficient conditions), then so does *CCMH*. Note that *MH* and *CCMH* coincide if every vertex of the *MH* frontier represents an observed DMU. The relationship between *MH* and *CCMH* can be further clarified by the fact that *MH* preserves the efficiency classification of both the observed DMUs and all non-observed production plans of the input-output space, whereas *CCMH* preserves the efficiency classification of the observed DMUs only.

It should be already evident that the standard static taxonomy of efficiency is available for the c-convex production sets along the lines of Section 2. Now how would imposing c-convexity instead of convexity influence the decomposition? As for structural efficiency, there is no difference whatsoever. This is simply because the condition C^{EP} cannot influence the convex combinations of DMUs lying on the congested or uneconomical part of the isoquant. That is, the backward bending boundaries of a convex set and a c-convex set are always exactly the same. However, for measuring technical and scale inefficiencies, the convexity assumption can have a dramatic effect. Obviously, an erroneously imposed convexity postulate can only decrease the measured overall "technical" inefficiency (OTE), i.e. $OTE=DF*STR*SCA$. Thus, erroneously specified convexity can both underrate overall efficiency, and distort the decomposition by overvaluing the technical and scale inefficiencies relative to the structural component. These effects of convexity should be taken into account in interpretation of results. The more general c-convexity property could help to remedy some of the potential specification errors.

Finally, note that it is possible to further relax our assumptions by applying c-convexity on the input or output isoquants, but not for the whole production set *T*. Following Bogetoft (1996), we could assume monotonicity of the production set, and apply c-convexity to inputs only. However, for sake of brevity we abstract here from these obvious alternative formulations.

5. EFFICIENCY MEASUREMENT

In this section we show that efficiency measures relative to c-convex production sets can be computed by Disjunctive Programming (DP). We also briefly discuss how the optimal solution to a DP problem can be inferred from optimal solutions to a series of Linear Programming (LP) problems akin to the standard DEA LP formulations.

In principal, *CCH* is obtained from *CH* by simply excluding all efficiency classification violating convex combinations. We can test whether any convex combination violates efficiency status of a weak efficient DMU *j* by solving the following Linear Programming test problem:

$$
\begin{aligned}
&\underset{q,l}{Max}\ q \\
&s.t.\ (1-q)x^{j} = Xl \\
&(1+q)y^{j} = Yl \\
&el = 1 \\
&l \in \Re_{+}^{w}
\end{aligned}
\tag{10}
$$

This test problem uses the graph measure by Briec (1997) as an instrument. Let q^*, l^* represent the optimal solution to problem (10). $q^* > 0$ indicates that the a convex combination of the reference DMUs $h = \{i \in S \mid l^{i*} > 0\}$ violates the efficiency classification. Consequently, that combination should be eliminated as infeasible. This elimination can be enforced by imposing to (10) an additional constraint

$$
l \in \bigcup_{i \in h} \{ l \mid l^{i} = 0 \}. \tag{11}
$$

That is, at least one of the current reference DMUs is excluded from the reference set by constraining the lambda weight equal to zero.

We can solve the test problem (10) again together with the constraint (11), and subsequently impose additional constraints until the optimal solution to the test problem equals zero. Hence, potential non-uniqueness of any reference set *h* does not matter. This procedure is repeated for all weak efficient DMUs. Let h_k denote the resulting set of indices $j \in S$ of DMUs that form an efficiency classification violating combination *k*, *k* = 1,…,*r*. Furthermore, denote the set of efficiency classification violating combinations by $H = \{h_1, \ldots, h_r\}$. Elimination of combinations $h \in H$ can be enforced by invoking a so-called *disjunctive constraint* (in the conjunctive normal form) written as

$$
l \in \bigcap_{h_k \in H} \left[\bigcup_{j \in h_k} \{ l \mid l^{j} = 0 \} \right].
$$

Incorporating this constraint to *CH* gives *CCH* an equivalent formulation to (7), i.e.

$$
CCH(X,Y) = \left\{ (x,y) \in \Re_{+}^{q+p} \left| \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} Yl \\ -Xl \end{pmatrix} \right. el = 1;\ l \in \bigcap_{h_k \in H} \left[\bigcup_{j \in h_k} \{ l \mid l^{j} = 0 \} \right];\ l \in \Re_{+}^{n} \right\}.
$$

Similarly, we can write *CCMH* equivalent to (8) as

$$CCMH(X,Y)=\left\{(x,y)\in\Re_{+}^{q+p}\left|\begin{pmatrix} y \\ -x \end{pmatrix}\leq\begin{pmatrix} Yl \\ -Xl \end{pmatrix}\right.; el=1; l\in\bigcap_{h_k\in H}\left[\bigcup_{j\in h_k}\left\{l \,\middle|\, l^{j}=0\right\}\right]; l\in\Re_{+}^{n}\right\}.$$

Finally, *RCCMH* can be written equivalent to (9) as

$$RCCMH(X,Y)=\left\{(x,y)\in\Re_{+}^{q+p}\left|\begin{pmatrix} y \\ -x \end{pmatrix}\leq\begin{pmatrix} Yl \\ -Xl \end{pmatrix}\right.; l\in\bigcap_{h_k\in H}\left[\bigcup_{j\in h_k}\left\{l \,\middle|\, l^{j}=0\right\}\right]; l\in\Re_{+}^{n}\right\}.$$

Computation of the standard efficiency measures relative to *CCH*, *CCMH* or *RCCMH* involves solving a DP problem for each DMU. For example, the Debreu-Farrell input measure for DMU *j* can be computed relative to *CCMH* as the optimal solution to the problem

$$\begin{aligned}
&\underset{q,l}{Min}\ q \\
&s.t.\ qx^{j}\geq Xl \\
&y^{j}\leq Yl \qquad (12)\\
&el=1 \\
&l\in\Re_{+}^{w} \\
&l\in\bigcap_{h_k\in H}\left[\bigcup_{j\in h_k}\left\{l \,\middle|\, l^{j}=0\right\}\right] \qquad (13)
\end{aligned}$$

The analogous *CCH* measure is simply obtained by substituting inequalities in (12) by equalities, while the *RCCMH* measure is obtained by deleting the constraint el = 1. Note that increasing (decreasing) returns to scale can be enforced by substituting this constraint by $el \geq 1$ ($el \leq 1$). Also alternative orientations or non-radial gauges can be computed relative to *RCCMH*, *CCMH* or *CCH* in a straightforward fashion.

Optimization theory knows a number of alternative finitely converging algorithms for solving Linear Programming problems with disjunctive constraints. See e.g. Sherali and Shetty (1980) for details. Let us suffice here to point some brief remarks on a basic relaxation principle. Note first that relaxing the disjunctive constraint (13) yields an ordinary DEA LP problem (12). The disjunctive constraint can be viewed to represent alternative sets of constraints, of which at least one must be satisfied. These constraints preserve the linear structure, enforcing lambda weights of the particular subsets of DMUs equal to zero. In fact, this is equivalent to excluding the corresponding subsets from the data matrices *X* and *Y*. These alternative data sets span a number of convex sets in the input-output space, which could be viewed as some kind of sub-technologies. For example, the *CCMH* set of Figure 1 can be presented as a union of three CMHs spanned by subsets (A,B), (B,C), and (C,D,E) respectively. A straightforward, albeit computationally hard way is simply to compute efficiency measures relative to all alternative sub-technologies i.e. by solving a relaxed version of LP problem (12). All these sub-technologies are feasible, so the minimum of the resulting optimal solutions is the optimal solution to the DP problem (12) with constraint (13). As a conclusion, the optimal solution to the DP problem can be inferred from optimal solutions to a series of LP problems.

The explicit characterization of c-convex technologies can be convenient for many 'predictive' purposes of DEA. However, it also entails a substantial computational burden. Computationally superior algorithms could be developed if we ignore the explicit characterization, and focus on solving efficiency measures as usual in DEA (see Kuosmanen (1999) for some heuristics). While this indeed seems an interesting avenue to explore further, we will abstract from this issue in the current paper and leave it for future research.

As for recovering shadow prices or approximating the elasticities of the frontier, c-convex reference sets provide equally fit frameworks as their convex counterparts, although the results can differ considerably. For simplicity, we have only considered the envelopment side (primal) problems here, but it is trivial to infer the optimal multiplier (dual) weights using the information of the optimal l vector, i.e. find the minimal dimensional hyperplane that supports all reference DMUs and the projected reference point of the evaluated DMU. In practice, one can solve the multiplier (dual) DEA formulation relative to the optimal reference set. Alternatively, it is possible to use the multiplier formulation throughout instead of (12), by modifying the data matrices instead of imposing restrictions on lambda weights, recalling that restricting a lambda weight equal to zero corresponds to excluding the corresponding DMU from the data. The multiplier weights can be used for example for computing elasticity of scale (5), or for inferring the qualitative (local) returns to scale properties. Moreover, the multiplier weights can be given a shadow price interpretation, which can be motivated by price endogeneity or uncertainty (see Kuosmanen and Post (1999b) for further discussion). Although recovering shadow prices is one promising by-product of the proposed model, its specific treatment falls beyond the scope of this paper, and so we leave it also for future research.

6. ILLUSTRATIVE EXAMPLE

We illustrate the practice of efficiency measurement relative to the c-convex reference sets by a simple example. For sake of brevity we focus on the *CCMH* set solely. For a more extensive application of the approach to a real-world data of 194 Finnish pesis batters (a game akin to baseball) with a single-input-three-output technology, see Kuosmanen (1999).

Consider the single-input single-output data of 5 DMUs given in Table 1, and illustrated by Figure 2. It is easy to see that DMUs #1, #2, #3, and #4 are efficient, and hence also weak efficient. The only inefficient one is the DMU #5. Now what would be the degree of inefficiency of this unit? In terms of the Debreu-Farrell input measure, *MH* efficiency amounts to 81.8%, while *CMH* efficiency is only 45.5%. Clearly, the convexity assumption makes a big difference. Also DMUs #2 and #3 would appear inefficient if convexity were imposed. Let us now compute efficiency measure of DMU #5 for the *CCMH* model.

Table 1: Example data set

	DMU 1	DMU 2	DMU 3	DMU 4	DMU 5
X	2	6	9	11	11
Y	2	3	6	11	5

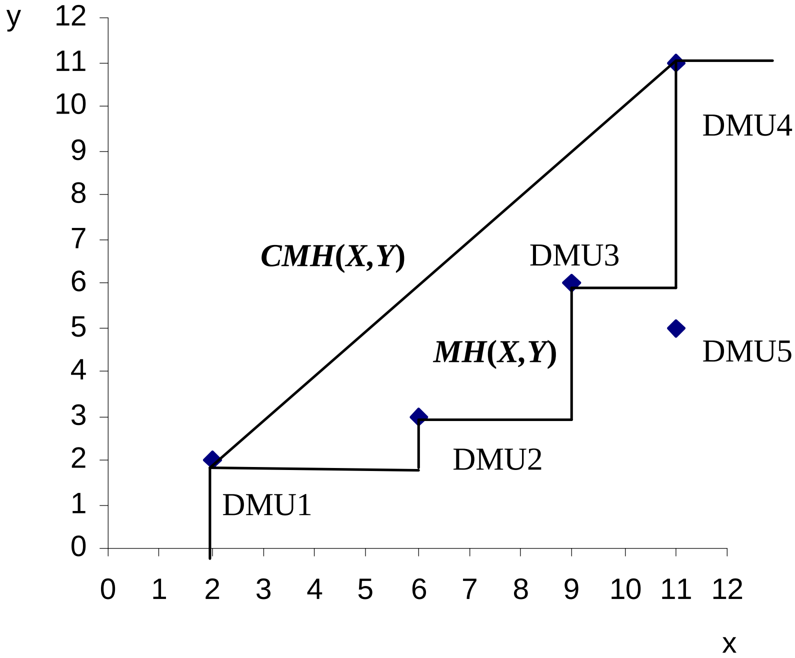

Figure 2: CMH and MH frontiers in the example data set

We start by characterizing the feasible convex subsets as explained in the previous section. We solve the test problem (10) for all efficient DMUs, excluding the ineficient DMU #5 from the reference data. For DMU *j*, $j=1,2,3,4$, the test problem reads:

Problem 6.1

$$\underset{q,l}{Max}\ q$$

$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$

$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$

$$l_1 + l_2 + l_3 + l_4 = 1$$

$$l_1, l_2, l_3, l_4 \geq 0$$

For DMUs #1 and #4 the optimal solution to this problem is $q=0$. For DMUs #2 and #3 the optimal solutions are: DMU #2:$q=1/3$; $l=\left(7/9, 0, 0, 2/9\right)$; DMU #3: $q=1/5$; $l=\left(37/45, 0, 0, 8/45\right)$. As for these two DMUs $q>0$, the convex combination of DMUs #1 and #4, which have lambda weights greater than zero in both solutions, must be eliminated by constraining $l_1 = 0$ or $l_4 = 0$.

We proceed by solving two new problems with restriction $l_1 = 0$ imposed in Problem 6.2, and restriction $l_4 = 0$ imposed in Problem 6.3. For DMU *j*, *j* = 2,3, the two test problems read:

Problem 6.2

$$\underset{q,l}{Max}\ q$$
$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_1 = 0$$

Problem 6.3

$$\underset{q,l}{Max}\ q$$
$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_4 = 0$$

The optimal solutions are the following. Problem 6.2: DMU #2: $q = 0$; DMU #3: $q \cong 0.0882$; $l = \left(0, \frac{57}{102}, 0, \frac{45}{102}\right)$; Problem 6.3: DMU #2: $q = 12/17$; $l = \left(\frac{9}{17}, 0, \frac{8}{17}, 0\right)$; DMU #3: $q = 0$. Elimination of the infeasible convex combinations is carried out by the following restrictions: $l_1, l_2 = 0$ or $l_1, l_4 = 0$ or $l_3, l_4 = 0$.

We proceed by formulating and solving the following three new problems corresponding to the three restrictions above:

Problem 6.4

$$\underset{q,l}{Max}\ q$$
$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_1, l_2 = 0$$

Problem 6.5

$$\underset{q,l}{Max}\ q$$
$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_1, l_4 = 0$$

Problem 6.6

$$\underset{q,l}{Max}\ q$$
$$s.t.\ (1-q)x_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$(1+q)y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_3, l_4 = 0$$

The optimal solutions to Problems 6.4 - 6.6 for DMUs #1 - #4 are all less than or equal to zero. Hence, the reference sets of Problems 6.4 – 6.6 characterize three convex monotone hulls that form CCMH as their union. Figure 3 illustrates the so-found convex sets. In that figure, CMH 1 corresponds to Problem 6.4, CMH 2 corresponds to Problem 6.5, and CMH 3 corresponds to Problem 6.6.

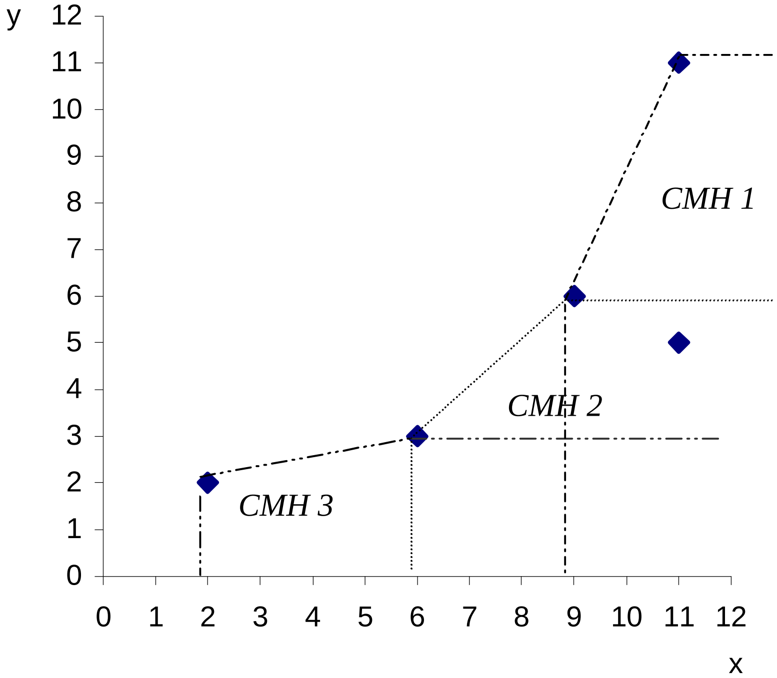

Figure 3: CCMH as the union on three convex subsets

Next, we can compute efficiency measures for inefficient units, say DMU #5, by solving the Debreu-Farrell measures relative to *CMH 1*, *CMH 2*, and *CMH 3* respectively, and consequently selecting the minimum of the feasible optimal solutions. For example, the efficiency score relative to *CMH 1* can be solved as the optimal solution to the LP problem:

Problem 6.7

$$\underset{q,l}{Min}\ q$$
$$s.t.\ qx_j \geq 2l_1 + 6l_2 + 9l_3 + 11l_4$$
$$y_j \leq 2l_1 + 3l_2 + 6l_3 + 11l_4$$
$$l_1 + l_2 + l_3 + l_4 = 1$$
$$l_1, l_2, l_3, l_4 \geq 0$$
$$l_1, l_2 = 0$$

Note that Problem 6.7 differs from Problem 6.4 only in that we have modified the objective function and the first two constraints so as to compute the Debreu-Farrell measure instead of the Briec graph measure. The corresponding DEA problems for computing Debreu-Farrell measures relative to *CMH 2* and *CMH 3* are modified accordingly from Problems 6.5 and 6.6 respectively, and are omitted here as obvious. For DMU #5, the optimal solutions are: *CMH 1*: $q = 9/11$; $l = (0,0,1,0)$; *CMH 2*: $q = 8/11$; $l = (0, 1/3, 2/3, 0)$; *CMH 3*: infeasible. The Debreu-Farrell efficiency measure for DMU #5 is then *Min*(9/11, 8/11) = 8/11. Like expected, this measure falls between the corresponding *CMH* (5/11) and *MH* (9/11) scores.

We solved the total of 17 LP problems (Problem 6.1 for 4 DMUs, Problems 6.2 and 6.3 for 2 DMUs, Problems 6.4 - 6.6 for 2 DMUs, and finally 3 LPs akin to Problem 6.7 for a single DMU). This small example gives a somewhat discouraging impression of the computational burden associated with the approach. Note that most of the computational cost was associated with characterizing the sub-technologies, i.e. *CMH*s *1*, *2*, and *3*. Once this has been done, computing efficiency scores for inefficient units has the marginal cost of 3 LPs only. Finally, this simple and straightforward relaxation strategy is not necessarily the most efficient algorithm for computing efficiency scores. If it is unnecessary to explicitly characterize the sub-technologies, an iterative branch and bound type of approach could reduce computational cost considerably. However, as noted in the previous section, the development of efficient computation codes is left for future research

7. CONCLUDING REMARKS

We proposed to relax the standard convexity property by invoking additional qualifications or conditions for feasibility of convex combinations. We especially focused on a condition that preserves the spontaneous Koopmans efficiency classification. We abbreviated the efficiency classification preserving conditional convexity as c-convexity. As both monotonicity and convexity imply c-convexity, but the converse does not generally hold, c-convexity can be viewed as the most general property of these three.

Substituting convexity by c-convexity, we constructed empirical DEA production sets as the smallest polyhedrons containing all DMUs, consistent with the imposed production assumptions. These c-convex production sets combine the attractive marginal properties of convex DEA technologies, allowing e.g. the measurement of scale elasticity, with preservation of the efficiency classification thus far only associated with the extremely non-convex *MH* model. Unfortunately, the general theoretical character of the c-convexity property does not fully carry over to empirical DEA approximations. Still, c-convexity provides a more general approximation than standard convexity even in small samples.

The distinct feature of the DEA formulations based on c-convexity is the disjunctive constraint that eliminates the infeasible, efficiency classification violating convex combinations. Consequently, solving the DEA model generally involves Disjunctive Programming. We briefly outlined a relaxation strategy for inferring the optimal solution from optimal solutions to a series of ordinary Linear Programming problems, using the fact that CCMH can be viewed as a union of convex monotone hulls of subsets of DMUs. This approach provides an explicit characterization of the production set, but is often computationally heavy. Development of a more efficient algorithm tailored for DEA provides an interesting challenge for future research.

Another promising avenue for the future research is to try to give the shadow prices of the c-convex frontiers a rigorous economic meaning. Price endogeneity and uncertainty, which have been almost ignored in this literature so far, provide a possible source of motivation for such an investigation.

We illustrated the computational formula by a simple single-input single-output example with 5 DMUs. Although the computational burden associated with this model is considerably higher than in the basic DEA models, the solutions are still obtained by a finite number of Linear Programming problems. We are quite convinced that the computational capacity currently available to researchers suffices to compute the proposed model even in relatively large-scale problems. It is also generally expected that the computational capacity can rapidly improve in the future. Therefore, development of models based on less restrictive maintained hypotheses appear affordable if the additional computational burden constitutes the only cost.

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